Formation, Assurance, and the Price of Commitment

Proofs and computational appendix

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This supplement accompanies the research note. Theorem and equation references refer to the reading edition unless stated otherwise.

A Proofs

A.1 Equal-treatment bound and the dependence calculation

Proof of Proposition 1. In any realized forming allocation, exactly ν selected investors receive a common $s \le V/\nu$. Selected participation implies $c_i \le s$ for each selected investor, so at least ν costs lie below V/ν. This argument does not require incentive compatibility. Independence gives the binomial count; independent heterogeneous indicators give the Poisson-binomial count. Under the posted contract, for any actions of other investors, acceptance produces either $s - c_i$ or zero. Its expected advantage has the sign of $s - c_i$ whenever the investor has positive probability of selection, and is zero otherwise. The specified strategies are weakly dominant and fill precisely the acceptable memberships. Universal refusal is an equilibrium because one investor cannot fill $\nu \ge 2$ seats and failed acceptance pays zero. □

Condition on $Z = z$ in Equation 4. Conditional indicators are independent Bernoulli with probability $q(z)$, giving Equation 5. At $\rho = 0$ use the binomial directly; at $\rho = 1$ all indicators coincide and the probability is π for $1 \le \nu \le n$. For interior ρ, the conditional fraction tends almost surely to $q(Z)$. Since $q(Z)$ has a continuous distribution, $\mathbb{P}(q(Z) = \alpha) = 0$, so bounded convergence gives $\mathbb{P}(q(Z) \ge \alpha)$, equal to Equation 6. The limit at independence and the boundary $\alpha = \pi$ cannot be substituted without treating the integer sequence separately.

Writing $a = \Phi^{-1}(\pi)$ and $d = \Phi^{-1}(\alpha)$, the argument of Φ in the limit has derivative

$$\frac{-a + d/\sqrt{1-\rho}}{2\rho^{3/2}}.$$

This proves Equation 7. The derivative in π is positive and that in α negative. At fixed ν a coupling adds another potential acceptance without removing old ones, so sufficient-demand probability weakly increases. The discrete backstop identity subtracts adjacent binomial tails. None of these probability arguments resolves subscription incentives.

A.2 Known-willingness reserve

Proof of Theorem 2. For $i \in A$, signing gives a strictly positive payoff whether the program forms or fails; refusal gives zero. Thus i signs with probability one at every type in any equilibrium. If A forms, formation follows. If A does not form but $A + j$ does, investor j strictly prefers signing once A signs: all supersets

of $A + j$ form and yield a strictly positive payoff. Once $A + j$ signs, every remaining investor also strictly prefers signing. This eliminates all strategies other than signing in at most three rounds of strict dominance. The argument holds under any common prior consistent with the pointwise payoff assumption and strictly increasing utility over the stated statewise gains.

If neither A nor any $A + j$ forms, let precisely A sign at every type. Its members strictly prefer their positive failure payoff. Every outsider can produce only failure by signing and obtains $\beta_i - h_i \leq 0$. Refusal is therefore a best response. This is a losing pure Bayesian equilibrium.

Let $r = \min_{F \in \mathcal{F}}(\sum_{i \in F} h_i - \max_{i \in F} h_i)$. Removing a member from a forming set cannot increase this objective when expenses are nonnegative, so a minimizer can be chosen inclusion-minimal. Fix such an F, choose $j \in F$ with largest expense, and set $\beta_i = h_i + \varepsilon$ for $i \in F \setminus \{j\}$, zero otherwise. The strictly assured set is $F \setminus \{j\}$, which does not form, whereas adding j forms. Its bonus sum is due in a possible failed profile and no other profile can owe more. Therefore $R = r + (|F| - 1)\varepsilon$.

For the converse, any implementing A contains, or can be enlarged by one investor to contain, an inclusion-minimal forming coalition F with at most one member outside A. If there is an outsider, call him j; otherwise choose any $j \in F$. The losing set $F \setminus \{j\}$ lies in A, so

$$R \geq \sum_{i \in F \setminus \{j\}} \beta_i \geq \sum_{i \in F \setminus \{j\}} h_i \geq \sum_{i \in F} h_i - \max_{i \in F} h_i \geq r.$$

If no singleton forms, the first expense inequality is strict because this losing set is nonempty and its members are strictly assured. Thus $R > r$ for every implementing contract. If a singleton forms, the zero-bonus contract implements formation: that investor signs strictly and everyone else follows. The reserve is zero and attained. □

A.3 Private-type screening and reserves

Proof of Theorem 3. Suppose the displayed conditions hold. For every investor with $b_i > 0$, his willing type signs strictly for every possible behavior of the others. At most one investor j remains; conditional on his type, independence implies that he faces success probability at least A_j. When $b_j \leq 0$, his willing payoff $Pg_j + (1 - P)b_j$ increases in P and is positive at $P = A_j$. Thus all willing types sign in every equilibrium. Every unwilling type now faces $P \geq A_i$. If $b_i \geq 0$, $-P\ell_i + (1 - P)b_i$ decreases in P and is strictly negative at A_i by the upper bound. If $b_i < 0$, both possible payoffs are negative. Hence all unwilling types refuse. The separating profile is itself an equilibrium, so it is the unique Bayesian strategy profile, including mixed behavior.

Conversely, exact screening in every equilibrium implies that separation is an equilibrium. Its two type incentive constraints give the weak lower and upper bounds. If at least two b_i are nonpositive, the investors with positive b_i can sign at every type and all others refuse. There are at least two refusers, so a single deviation cannot complete the cohort. This is another equilibrium.

It remains to rule out equality. At an upper-bound equality for investor i, let his unwilling type sign with a small positive probability and keep every other strategy separating. His own success probability remains A_i, leaving him indifferent. Other unwilling types face weakly higher success probabilities; under their weak upper constraints they continue to prefer refusal. Other willing types either have positive b, making signing

strictly preferable, or have nonpositive b, in which case the higher success probability only helps. This constructs another Bayesian equilibrium. Therefore every upper bound must be strict.

Now suppose a lower bound binds. It is strictly negative, so it can concern only the single investor j with $b_j \leq 0$. Let his willing type refuse with a sufficiently small positive probability. All other willing types have $b_i > 0$ and continue to sign strictly. Investor j's own success probability remains A_j, so his willing type remains indifferent and his unwilling type refuses. The other unwilling types' strict incentive constraints survive a sufficiently small perturbation, because the population is finite and the payoff expressions are continuous. This gives another equilibrium. Every lower bound must therefore also be strict. □

Proof of Corollary 4. For an indispensable cohort, every proper subset can be the failed signing set. Nonnegative bonuses imply $R(\beta) = \sum_i \beta_i - \min_i \beta_i$. In an implementing contract all but at most one investor have $\beta_i > h_i$. The possible exception j must satisfy $\beta_j \geq 0$ and $\beta_j > h_j - A_j g_j/(1 - A_j)$, giving the componentwise lower limit $v^{(j)}$. If all investors are strictly assured, designate any j; its lower limit is still no larger. The liability function is coordinatewise nondecreasing, establishing the lower bound in Equation 11.

For any j, choose $\beta_i = h_i + \varepsilon$ for $i \neq j$. Choose $\beta_j = 0$ if its lower endpoint is strictly negative; otherwise choose it just above that endpoint, with an arbitrarily small positive adjustment when the endpoint is zero. Since each upper bound exceeds zero in net terms, sufficiently small positive adjustments satisfy it for the assured investors. The exception's net lower endpoint is negative, so his upper constraint is slack. If $h_j = 0$ and a small positive adjustment makes him strictly assured, the condition allows this too. Thus these contracts approach the displayed bound.

To check nonattainment, use the proper subset obtained by dropping a smallest component of $v^{(j)}$. Its sum equals the limiting liability. If this subset contains an assured investor, that investor's gross payment strictly exceeds his limiting expense, giving a strict increase. The only remaining case is $m = 2$ and the dropped member is the sole assured investor. The included exception then has $a_j > 0$, unless both limiting components are zero. For $a_j > 0$ his lower incentive constraint requires a strict increase. If both are zero, any strictly assured investor has a positive bonus and therefore a positive liability. Every implementing reserve is strictly above the infimum. □

Proof of Theorem 5. Under separating play, each other investor has a willingness probability strictly between zero and one. For every i, independence and $n \geq \nu \geq 2$ imply strictly positive failure probability f_i and strictly positive selection probability a_i^{sel} after he signs. Full reimbursement makes an unselected signature's net payoff zero. If $b_i > 0$, types $x_i \in (-f_i b_i / a_i^{\text{sel}}, 0)$ strictly prefer signing. Such types occur with positive probability by the continuous-margin assumption. Exact screening therefore requires $b_i \leq 0$ for all i. Given universal refusal, signing cannot form and produces at most zero, so universal refusal is a Bayesian equilibrium. Yet sufficient willing demand has positive probability. This contradicts all-equilibrium formation whenever such demand exists. □

A.4 Interim participation and the uniform threshold

Proof of Proposition 6. Efficient selection is nonincreasing in an investor's reported cost. Standard threshold payments implement it in dominant strategies and normalize highest-type utility to zero. The one-

dimensional envelope identity gives Equation 14 for every BIC implementation. Fubini's theorem yields

$$\mathbb{E}\int_{c_i}^{\bar{c}_i} q_i(t)\,dt = \int_0^{\bar{c}_i} F_i(t)q_i(t)\,dt = \mathbb{E}\left[\frac{F_i(c_i)}{f_i(c_i)}\mathbf{1}\{i \text{ selected}\}\right] = I_i.$$

Payments equal selected costs plus utility, so expected total payments less distributable value are $I-W+\sum_i u_i$. Since $u_i \geq -\kappa_i$, expected nondeficit requires $\sum_i \kappa_i \geq I-W$. Nonnegativity gives the positive part. To attain it, subtract constants $k_i \in [0, I_i]$ from canonical payments with $\sum_i k_i = [I-W]_+$. Such constants exist because $W \geq 0$. They leave reporting incentives unchanged, set $u_i = -k_i$, and give ex-ante utility $I_i - k_i \geq 0$. If $I < W$, canonical payments already satisfy expected nondeficit. This construction is not a statewise financing theorem. □

Proof of Theorem 7. Fix $\nu \geq 2$, $V = \nu/2$, and put $T = c_{(\nu)}$. Conditional on $T = t$, the sum of the selected costs is distributed as tY, where $Y = 1 + \sum_{j=1}^{\nu-1} U_j$ and the U_j are independent uniform variables. The density of Y is symmetric and unimodal about $(\nu+1)/2$, on $[1, \nu]$. Define

$$g(t) = \mathbb{E}[(2tY - V)\mathbf{1}\{tY \leq V\}].$$

For $t < t_0 = V/(\nu+1)$, truncation is inactive and $g(t) = (\nu+1)t - V < 0$. For $t > t_0$ with $t < V$, put $w = V/t < \nu+1$. Symmetry and unimodality give $\mathbb{E}[(Y-w/2)\mathbf{1}\{Y \leq w\}] > 0$: extend the density by zero outside $[1, \nu]$ and pair points around $w/2$. For $x > 0$, symmetry and unimodality give $f_Y(w/2+x) \geq f_Y(w/2-x)$ because the right point is weakly closer to the density's center. The unmatched support contributes a strictly positive interval when $1 < w < \nu$; when $w \geq \nu$, the untruncated expectation is $(\nu+1-w)/2 > 0$. For $w \leq 1$ the integral is zero. Consequently g has exactly one sign change, from negative to positive, and may be zero at the far right.

Apart from a positive normalizing constant, $D_{n,\nu}(V)$ is the integral of $g(t)t^{\nu-1}(1-t)^{n-\nu}$ on $[0,1]$. If that integral is nonpositive for some n, its reweighting by $1-t$ is strictly negative: subtract $(1-t_0)$ times the old integral to obtain the integral of $(t_0-t)g(t)$ against the old positive weight, which is strictly negative. Thus the sign crosses at most once as n increases.

At $n = 2\nu+1$, the untruncated expectation is

$$2\mathbb{E}S_\nu - V = \frac{\nu(\nu+1)}{n+1} - \frac{\nu}{2} = 0.$$

Removing $S_\nu > V$, a positive-probability event with positive integrand, makes D strictly negative. It remains to prove $D_{2\nu,\nu}(\nu/2) > 0$. For $2 \leq \nu \leq 127$, the rational formula in Appendix B gives positive integer numerator and denominator for every case; all certificates accompany the code.

For $\nu \geq 128$, $\mathbb{E}S_\nu = \nu(\nu+1)/[2(2\nu+1)]$. Changing one of the 2ν independent uniform costs changes the sum of the lowest ν by at most one. The bounded-differences inequality (McDiarmid, 1989) therefore gives

$$\mathbb{P}(S_\nu > \nu/2) \leq \exp\left[-\frac{\nu^3}{4(2\nu+1)^2}\right] \leq e^{-\nu/17}.$$

The second inequality holds for $\nu \geq 17$. Since $0 \leq S_\nu \leq \nu$,

$$D_{2\nu,\nu}(\nu/2) \geq \frac{\nu}{2(2\nu+1)} - \frac{3\nu}{2} e^{-\nu/17} > 0.$$

For the last inequality, $3(2\nu+1)e^{-\nu/17}$ is decreasing for $\nu \geq 17$ and is less than .415 at $\nu = 128$. Positivity at 2ν and the single-crossing result prove the first-crossing claim. Finally, with one seat and one investor, $\int_0^{1/2}(2c - 1/2)dc = 0$, so the threshold is already one. □

A.5 The constrained closing-frequency frontier

For any BIC allocation with uniform costs, the envelope identity requires expected virtual payments $\mathbb{E}\sum_{i \text{ selected}} 2c_i$, up to the sum of highest-type utility constants. If total negative interim allowance is at most k, expected nondeficit therefore requires

$$\mathbb{E}\left[\left(2\sum_{i \text{ selected}} c_i - V\right)\mathbf{1}\{\text{form}\}\right] \leq k.$$

Replacing a selected coalition by the cheapest ν lowers this virtual deficit without reducing closing frequency. The relaxed pointwise maximization with multiplier $\lambda \geq 0$ forms when $1 - \lambda(2S_\nu - V) > 0$, also imposing $S_\nu \leq V$. Thus a cost cutoff is optimal. This cutoff allocation is monotone in each own cost, so threshold payments attain the relaxed bound; constant enrollment charges up to k preserve ex-ante participation because selected real surplus is nonnegative. If $D(V) \leq k$, all efficient draws form. Otherwise the unique relevant root $D(s) = k$ lies above $V/2$, where D is increasing. For two investors, S_2 has density s on $[0, 1]$, yielding Equation 19 by integration.

B Computational appendix

B.1 Exact order-statistic integration

Set $k = \nu - 1$, $m = n - \nu$, $C = n!/(k!m!)$, and $u_j = \min\{1, V/(j+1)\}$. For integer $r \geq 0$, define

$$J_{m,r}(d, u) = \int_0^u (1-t)^m (V - dt)^r \, dt.$$

Using the Irwin–Hall distribution of $\sum_{j=1}^k U_j$, exact integration gives

$$\mathbb{P}(S_\nu \leq V) = C\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{J_{m,k}(j+1, u_j)}{k!}, \tag{20}$$

$$D_{n,\nu}(V) = C\sum_{j=0}^{k}(-1)^j \binom{k}{j} \left[\frac{V J_{m,k}(j+1, u_j)}{k!} - \frac{2J_{m,k+1}(j+1, u_j)}{(k+1)!}\right]. \tag{21}$$

Every term is rational at rational V. If $u = V/d \le 1$, expand $(1 - uv)^m$ after $t = uv$:

$$J_{m,r}(d,u) = V^r u \sum_{a=0}^{m} (-1)^a \binom{m}{a} u^a \frac{a!r!}{(a+r+1)!}.$$

If $u = 1 < V/d$, expand $(V - dt)^r$ instead:

$$J_{m,r}(d,1) = \sum_{a=0}^{r} \binom{r}{a} V^{r-a}(-d)^a \frac{a!m!}{(a+m+1)!}.$$

These formulas, evaluated with arbitrary-precision rational numbers, produce the finite certificates for Theorem 7. The verifier asserts positive numerator and denominator at $n = 2\nu$ for all $2 \le \nu \le 127$. A second integration method and a separate review check the construction. The certificates establish those finite signs; the analytic tail argument establishes every remaining integer.

B.2 Formation structures, private types, and numerical precision

The verifier recursively enumerates all monotone Boolean functions on labeled investors. Splitting on the last investor gives two monotone functions a, b on one fewer investor, constrained by $a \le b$ pointwise. Excluding the constantly failing and constantly forming functions leaves 1, 4, 18, 166, and 7,579 nontrivial structures for one through five investors. Five expense schedules include all-zero expenses, equal positive expenses, increasing and decreasing heterogeneous expenses, and mixed zero-positive expenses. All 38,840 structure-schedule checks compare the formula against direct optimization over strictly assured subsets. Pure profiles are enumerated independently of the displayed implementing-set criterion. Mixed and Bayesian implementation follows from the proof, not from pure enumeration.

For private willingness, 1,350 finite-type checks cover separating interiors, upper-bound violations, and two weakly assured investors, with two to four investors and exact rational payoffs. Explicit mixed witnesses cover both boundary equalities. At $p_1 = p_2 = 1/2$, $g_i = \ell_i = 1$, and $b = (1, 1/2)$, investor 1's unwilling type may sign with probability .4 while both willing types sign and investor 2's unwilling type refuses. The first unwilling type remains indifferent, while the second strictly refuses. At $b = (-1, 1/4)$, investor 1's willing type may sign with probability .8, with the remaining types separating. These cases reject an implementation certificate based only on the intended pure profile.

Gaussian finite-pool probabilities are computed by adaptive integration and independently by a dense midpoint rule over factor quantiles. Four fixed-seed simulations use 150,000 draws each and report their binomial standard errors. Exact order-statistic results are also checked against direct integration in the two-seat case. Separate fixtures distinguish unequal tickets, funding shortfalls, payment caps, and expiry. The experiments do not treat a numerical participation grid as a contractual lot size. Exact identities use rational arithmetic; floating-point probability checks have explicit tolerances. The flagship's \$25-per-investor and \$100-aggregate CE stability limits remain release limits for any dollar-valued approximation; no new such approximation is used here.

B.3 Reproduction and evidence status

The accompanying package records the Python and dependency versions, fixed seeds, script hashes, frozen results, source access, and review dispositions. Reproduction runs without a model API or network access. Every numerical table is linked to a saved result. Earlier editions are preserved by file hashes. The original proposal's combined "attaining contract" is not released: its known-willingness assurance assumption does not fit the continuous private-cost demand model. The restricted impossibility and finite-gap characterization replace that claim. Real-data forecasting, timed arrivals, optimal closing windows, and simultaneous raises remain separate research projects.

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