The Formation Game

Commitment and the Investment Frontier

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Working paper October 2026

Abstract

In portfolio theory each investor has a choice set of his own, within which a fund is a convenience whose value falls as separate accounts become cheaper. Many investments, however, exist only if several investors commit to them, so that the choice set is the outcome of a game. Funds combine committed capital, managerial discretion and common exposure; this paper asks which of these must be shared for such investments to form, which is to ask which terms of a fund cannot be written account by account, and what keeping the others separate costs. Under simultaneous commitments, robust formation requires a promise to pay in the state of failure whose least size the paper computes over all terms contingent on who commits, and an escrow approaches it. Investors who keep separate exposures can finance a set of investments together: differences in mandates diversify like returns, a price on crowded positions absorbs idiosyncratic imbalances at a bounded expected cost, and the right to decide after that price is known requires capacity of the order of the square root of the number of investors, under every efficient mechanism with dominant-strategy incentives and for any reliability above one half, whereas participation that binds before the imbalance is known, with the premium returned within the book, needs a bounded amount. Common exposure economizes on records and incentive pay; it is not what makes investments form. Better separate accounts compress the equilibria of unassured investments, so that investments that still create surplus can disappear.

Keywords: collective commitment; contingent capacity; discretion; investor mandates; robust implementation; replacement capital; separate accounts; collective search.

JEL: G11, G23, G24, C72, D47, D82, D86.

1 Introduction

Investment funds combine committed capital, managerial discretion and common portfolio exposure. This paper asks which of these functions must be shared for investments to form, and what it costs investors to keep the others separate. Portfolio choice takes the available investments as given: Markowitz (1952) gives each investor a set of portfolios and asks which he should hold, and the separation theorems state when a few common funds serve investors whose preferences and circumstances differ, so that pooling is the cheapest way to hold what everyone would choose anyway (Tobin, 1958; Cass and Stiglitz, 1970; Ross, 1978). That reason for pooling is a matter of cost, and the cost of an account kept for a single investor, tailored to his taxes, liabilities, exclusions and existing holdings, falls with the cost of interpreting mandates, reconciling records and maintaining an attributable history of holdings. Many investments, however, and most private ones, do not exist until several investors commit to them, so that the set from which an investor chooses depends on what others choose, and the investment frontier he faces is the outcome of a game rather than a datum. Under the three heads of committed capital, discretion and common exposure, a fund that forms such an investment holds five things in common: it supports the commitments on which formation depends, so that the investment forms although each investor would refuse if he expected the others to refuse; it delivers principal whose permissions allow it to finance what the investment requires; it binds each investor before the investments and their terms are known, so that no one can withdraw once they are; it gives a manager discretion to place that principal; and it gives every investor the same proportional exposure to whatever results.

The answer is that commitment support, compatible principal and discretion must be shared, in amounts the paper computes; that the moment at which participation binds sets the price of the capacity a closing needs; and that exposure need not be shared. The argument has two parts. Commitment itself requires, under simultaneous commitments, support of a size the paper computes, whose cheapest robust form is a promise to pay in a state in which the investment does not exist. A set of investments that a population finances together requires principal whose permissions cover the set and discretion whose amount has the structure of diversification, and the paper studies it as a *book*: a set of investors who commit to finance a set of projects as a single event while keeping their own exposures, in which places in a crowded project carry a price and the premium so collected is either retained or returned. On the moment at which participation in a book binds the paper's central result turns. When investors keep the right to refuse until the book's places and prices are known, the capacity a closing draws on grows with the square root of the number of investors under every efficient mechanism with dominant-strategy incentives, whatever reliability above one half is required; when participation binds before the common imbalance is known and the premium is returned within the book, a bounded reserve suffices. Common exposure is not required, since it affects whether investments form only through which investors are willing to commit, and what it saves is the cost of records and of incentive pay. Where mandates differ idiosyncratically the book's cost of discretion is bounded while a pooled fund's grows with the population, and the theory predicts the book; where a common component

moves mandates together, the discretion required is a fixed share of capital however many investors there are and its cost, like the cost of the records and incentive pay that a fund economizes, is proportional to the population, so that those economies can then decide for the fund. Four rules for a book follow: bind participation before the common imbalance is known, price the crowded position, return the premium within the book, leaving out each investor's own report where exact incentives are required, and reserve replacement capital to a quantile of the shortfall, which under a retained veto is of order $\sqrt{n}$ and under binding enrollment a few invitations.

Let an investment form when the investors who commit to it constitute a coalition in a family $\mathcal{F}$ that is closed under enlargement, so that a count of seats, a threshold of capital and a set of required roles are special cases, and let h_i be investor i's expense of standing ready, which he loses if he commits and the investment fails. When commitments are simultaneous and willingness is known, the outcome in which nobody commits is an equilibrium whatever investors would gain. Let

$$R^{*} = \min_{F \in \mathcal{F}} \Big(\sum_{i \in F} h_i - \max_{i \in F} h_i \Big), \tag{1}$$

the smallest, over coalitions able to form the investment, of the sum of their members' expenses less the largest of them. Payments that depend only on each investor's own commitment remove the equilibrium of refusal by transfers on the equilibrium path of more than R^{*} unless a single investor can form the investment alone, while payments conditioned on failure remove it with nothing paid in equilibrium and a liability whose infimum is R^{*} (Theorem 3), and among all payment schemes that depend on which investors commit, whatever they pay when the investment forms, none makes formation the only rationalizable outcome with a liability of R^{*} or less (Theorem 4). The literature on contracting with externalities shows that terms contingent on others' choices can make participation the unique outcome at no cost on the equilibrium path (Segal, 2003; Winter, 2004; Halac, Kremer and Winter, 2020), and the theorem prices what such terms require off it. The bound survives, for closed cohorts and counts of seats, when formation need only be the outcome of every correlated equilibrium, and that weaker standard can lower it for other structures but never below the combined expenses of the cheapest investors, one fewer in number than the smallest forming coalition (Theorem 5); cheaper schemes that rely on investors coordinating leave failure rationalizable (Proposition 7). The capacity is never drawn when willingness is known, so that one promisor can back many investments with the largest single requirement rather than their sum (Proposition 8); a public order in which earlier commitments bind needs no capacity among investors who believe in one another's future rationality (Proposition 9), and an escrow released as each public round closes needs only the largest round's requirement, so that a calendar divides the promise as well as shortening it (Theorem 11). When willingness is private, an escrow attracts investors who would rather the investment failed, and when unwilling investors can be arbitrarily close to indifference it screens them out only when paired with a charge or placed within an order (Theorem 13).

The second part begins by separating commitment from exposure. When a book of investors commits to finance a set of projects as a single event, the least support that makes its closing

the unique rationalizable outcome depends on exposure only through which investors are willing to commit (Proposition 17), and what the book needs is compatible principal: the projects can be financed exactly when every set of them is covered by the capital permitted to finance it, so that aggregate capital overstates a population's capacity to invest whenever permissions are incompatible (Proposition 16). The discretion needed to absorb the mismatch between what investors permit and what projects require has the structure of diversification. When restrictions differ idiosyncratically, the flexible capital needed grows only with the square root of the number of investors, so that with ten thousand investors and two projects about two and a half percent of the capital suffices for 99 percent reliability, while a common component of mandates, through a shared view, adviser or recommendation, requires a fixed share however many investors there are (Theorem 18). Permissions are thus a dimension of diversification distinct from returns, and their systematic part determines how much discretion must be held in common.

Discretion has a price, and the mechanism that charges it matters. An efficient book moves only the investors needed to balance its projects, those who least mind being moved, and the Vickrey–Clarke–Groves mechanism, which with two projects is a uniform price on places in the crowded project, implements that assignment in dominant strategies. When mandates are idiosyncratic, the expected total cost of the discretion the book uses stays bounded as the book grows and converges to $(m-1)/(4\varphi g_c(0))$ with m projects, a share φ of investors who accept reassignment and a density $g_c(0)$ at zero of the cost of being moved, while pro rata exposure imposes on the average investor a cost that does not shrink (Theorem 19). The flexible capital needed to finance a book is of order $\sqrt{n}$, while the flexible population needed to finance it cheaply is of order n, so that a pool of discretion should be broad and shallow, granted by many investors and used on few. Such a book finances every project that a blind pool finances and admits investors whom the pool cannot, while separate consent finances them all with a probability that tends to zero (Corollary 21), and paying investors for flexibility invites them to declare the crowded project in order to be paid for leaving it, whereas pricing the crowded project leaves no such incentive (Proposition 20).

The two parts meet in the question of which investors remain willing once a book's terms are known, and three results measure what the right to decide later costs, in a benchmark with two equal projects and every investor flexible. First, if investors keep a veto until places and prices are announced, the investors who refuse are those who pay the premium for a place in the crowded project and whose margins fall short of the price, and their number moves with the one imbalance that all investors share, because a common clearing price turns independent differences in fit into correlated decisions to participate; the shortfall is of order $\sqrt{n}$ and fluctuates on that scale, so that at critical load the probability of formation tends to zero and replacement capital proportional to $\sqrt{n}$ buys only a confidence level (Theorem 22). Second, if investors bind themselves before favored projects and costs are known, the expected shortfall is the same to first order but it concentrates, so that replacement capital need cover its mean rather than its tail (Corollary 23). If the premium is also returned within the book, each investor's expected net burden is his share of the bounded real cost of discretion, the shortfall stays bounded, and a book in which

every investor is indispensable forms with probability tending to $e^{-\mu_v}$, where μ_v is the density of participation margins at zero times the limit of the expected real cost of the book's discretion; the conclusion survives when investors know their own types at enrollment (Theorem 22), and a rebate leaving out each investor's own report preserves it while making truthful reporting dominant, at the cost of tripling μ_v (Proposition 24). Third, no other pricing of the places escapes the veto while favored projects are private: under every mechanism that makes the efficient assignment with dominant-strategy incentives, the capacity a closing draws on, whether replacement capital for the places refused or cash behind the mechanism's own payments, is of order $\sqrt{n}$ in expectation and for any reliability above one half, a threshold that is exact (Propositions 25 and 26). The threshold rests on the privacy of favored projects, since with mandates that a register can verify, a procurement auction that pays the investors moved needs no places and bounded cash whatever veto investors retain (Corollary 27); and it rests on incentives, since a mechanism that pays each investor the expected externality of his report keeps the shortfall bounded under truthful reports, which the veto undoes by turning a payment for claimed flexibility into an option. A bounded real cost, accepted before the common shock is known, thus leaves a probability of formation bounded away from zero, whereas the same book with a retained veto fails because transfers of total order $\sqrt{n}$ fall, after that shock, on one side of it. A narrow mandate and a late decision are the two forms of control an investor can keep, and the first is accommodated by flexible capital, the second by replacement capital.

The remaining results identify what limits the gains. Common exposure economizes on incentive pay and on records: with complete records, pay that depends on each account's own record can make investigation a manager's best choice (Theorem 29), but pay written on pooled records can make it much cheaper, and paying a manager together is itself a formation problem (Propositions 44 and 45); a book that offers k exposure profiles loses, relative to individually optimal holdings, the tail of the spectrum of investors' risk-adjusted optimal exposures, so that the number of profiles it should offer rises as the cost of a record falls (Proposition 46). Individual terms can cost more than individual exposure, because equal-treatment clauses make terms common: securing a pivotal commitment costs, under rate matching, his shortfall times the ratio of the commitments entitled to his terms to his own (Proposition 47), and renegotiating the clauses by waivers is itself a formation problem (Proposition 48). Discretion should not absorb imbalances that reflect knowledge rather than fit (Proposition 28). Better separate accounts act through willingness: under a property the paper calls *completion*, raising one investor's outside value by Δ raises welfare in the worst equilibrium by Δ and changes it in the best by between $\Delta - G$ and Δ, where G is the range between them (Theorem 32), so that the most assurance can add shrinks as outside options improve (Corollary 33), while a wider pool reached by search leaves the floor unchanged and weakly raises the best equilibrium (Proposition 57). Search acts on every quantity the two parts identify, and not always in the same direction: finding complementary permissions and compatible substitutes lowers the shortfall of principal and the commitment support a book needs, while search that leads many investors to the same recommendation creates the systematic component of mandates that makes discretion expensive (Proposition 43).

The institutions that assemble shared investments use the instruments of the theory, and the comparison raises a puzzle. Under known willingness and an assumed expense of standing ready of one percent of each commitment, the carry of the capacity that eliminates strategic nonformation in a $300 million fund of 30 investors runs from half a basis point to about ten, depending on who holds it, yet practice rarely offers escrows above expenses; it relies instead on refunds, on orders where earlier commitments bind, on anchors, and on leads whose term sheets do not bind and whose promises rest on reputation (Section 7). The escrow and the individually permitted book are in this sense design proposals, and the paper sets out the measurement that would test the second, which compares it with funds that already individualize through excuse rights, side letters and co-investment.

Related literature. The paper joins the theory of the firm's boundaries (Coase, 1937) to the theory of portfolio choice. Separation and spanning results explain when heterogeneous investors can be served by mixtures of common funds (Tobin, 1958; Cass and Stiglitz, 1970; Ross, 1978); Mamaysky and Spiegel (2001) derive the objectives of funds from the heterogeneous trading needs of investors who cannot trade continuously, Acharya and Bisin (2005) characterize optimal market structures by the dispersion of participants' exposures to the traded assets, security design treats the set of traded claims as endogenous (Duffie and Rahi, 1995; Boyd and Prescott, 1986), and Merton and Bodie (2004) study how institutional forms adapt to the costs of financial services. Funds have been explained by diversification in delegated monitoring (Diamond, 1984), by the indirect sale of information (Admati and Pfleiderer, 1990), by liquidity insurance (Diamond and Dybvig, 1983; Jacklin, 1987) and, for private equity, by the agency problems that ex ante pooling disciplines (Axelson, Strömberg and Weisbach, 2009); the question here is which of the things a fund holds in common its functions require.

The closest antecedent on formation is Akerlof and Holden (2019), who treat capital assembly for projects with increasing returns as an equilibrium problem. Threshold provision of public goods supplies the strategic structure (Palfrey and Rosenthal, 1984; Bagnoli and Lipman, 1989; Admati and Perry, 1991; Marx and Matthews, 2000), and all-or-nothing crowdfunding applies it to project finance (Strausz, 2017; Ellman and Hurkens, 2019; Chemla and Tinn, 2020). Assurance contracts and refund bonuses use payments upon failure, and the dominant assurance contract already pays nothing in equilibrium (Tabarrok, 1998; Zubrickas, 2014; Cason and Zubrickas, 2017); the divide-and-conquer literature characterizes the least costly contracts that make participation the unique outcome when agents impose externalities on one another (Segal, 1999, 2003; Winter, 2004; Bernstein and Winter, 2012), and Halac, Kremer and Winter (2020) let a firm's own capital insure its smallest investors and observe that contracts contingent on third parties' choices would extract the full surplus. Theorems 4 and 5 bound, over all profile-contingent schemes, the liability off the equilibrium path that makes such contracts work, under rationalizability (Bernheim, 1984; Pearce, 1984) and under the uniqueness of correlated equilibrium (Aumann, 1974, 1987; Kajii and Morris, 1997), whose gap with Nash implementation is studied by Bergemann, Morris and Tercieux (2011) and whose fragility follows from Weinstein and Yildiz (2007) and Oury and Tercieux (2012).

Sequencing lowers the cost of unique implementation (Winter, 2006, 2010), announced leads remove the equilibrium of refusal (Andreoni, 1998), and early refund bonuses lower the worst-case payout of a campaign (Cason, Tabarrok and Zubrickas, 2021); Theorem 11 gives the saving from a limited calendar under common belief in future rationality (Perea, 2014; Penta, 2012). The compatibility condition is Gale's feasibility theorem for flows and Hall's condition (Gale, 1957; Hall, 1935). The square-root law of Theorem 18 is that of inventory pooling and of staffing at critical load (Eppen, 1979; Halfin and Whitt, 1981), the persistence of a common component that of pooling under dependent demand (Corbett and Rajaram, 2006), and the value of a little flexibility is known from process flexibility and resource pooling (Jordan and Graves, 1995; Bassamboo, Randhawa and Van Mieghem, 2012; Tsitsiklis and Xu, 2012); Theorem 19 uses the assignment game and its Vickrey–Clarke–Groves prices (Vickrey, 1961; Shapley and Shubik, 1971; Leonard, 1983; Demange, Gale and Sotomayor, 1986) and confronts the impossibility of efficient, incentive-compatible and balanced mechanisms (Green and Laffont, 1977). Theorem 22 belongs to the analysis of participation constraints in mechanism design, in which participation decided before private information arrives is compatible with efficiency and budget balance (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979) while participation decided afterwards can preclude it (Myerson and Satterthwaite, 1983; Cramton, Gibbons and Klemperer, 1987), and to its large-population branch (Gul and Postlewaite, 1992; Mailath and Postlewaite, 1990; Hellwig, 2003; Battaglini and Palfrey, 2024), and its rebate that leaves out each investor's own report is the redistribution of Vickrey–Clarke–Groves payments studied by Bailey (1997), Cavallo (2006), Moulin (2009) and Guo and Conitzer (2009); in a book the real cost is bounded, and what determines whether formation survives is whether, while investors can still refuse, their net burdens move with a common shock. Reputation can make informal promises credible within the limits set by the future surplus of a relationship (Levin, 2003), which bears on the nonbinding leads observed in practice. Completion is weaker than core implementation and requires no refinement, and the willing-membership construction behind it underlies Moulin mechanisms for cross-monotonic cost sharing (Sprumont, 1990; Moulin, 1999; Moulin and Shenker, 2001; Juarez, 2013); the equal-treatment multiplier is a financial counterpart of the commitment effect of most-favored-customer clauses (Cooper, 1986). Theorem 29 applies the detectability characterization of moral hazard (Legros and Matthews, 1993; Fudenberg, Levine and Maskin, 1994; Rahman, 2012) to signals that are the accounts' marginal records, and Proposition 44 the minimum-likelihood-ratio result for risk-neutral agents with limited liability (Innes, 1990; Demougin and Fluet, 1998). Shahidi et al. (2025) ask how artificial agents that search, communicate and contract at low cost change market design. Evidence on direct investing and co-investment documents investors combining delegated and direct exposure in private equity (Fang, Ivashina and Lerner, 2015; Lerner et al., 2022; Braun, Jenkinson and Schemmerl, 2020), and Begenau and Siriwardane (2024) document fee tiers within private equity funds.

Section 2 sets out accounts, records and the three kinds of term. Section 3 treats commitment support, and Section 4 the book: compatible principal, discretion and its price, and the formation of the book itself. Section 5 prices what individual exposure and individual terms cost instead,

Section 6 develops what better outside options do to investments that need several investors, and Section 7 turns to institutions and measurement. Section 8 collects the limits of the analysis and concludes. Appendix A contains the proofs, and Appendix B the results and examples that the main text summarizes: screening under private willingness beyond the theorem stated, the limit of information, escrow under private information and risk aversion in a closed cohort, the burden a book places on its investors and search on three margins, the economy behind the delegation results and the two delegation propositions, exposure profiles and equal treatment, threshold programs, divisible contributions, completion and complementarity, continuous portfolios, the boundary of the compression theorem and excess separation, collective search, incidence and the technology economy, the sensitivity analysis behind the comparison of Section 7, and one documented transaction.

2 Accounts, records and terms

2.1 Investors, accounts and outside values

There are finitely many investors, $i \in N$. An investor is a holder of capital with its own decision authority: an individual, a trust, a family office, an endowment, or a fund acting as a single participant. Multiple accounts of the same economic principal do not create multiple budgets. Each investor acts under a mandate, which states objectives and hard investment constraints, and under contracts, which create obligations; authority to commit his capital is his to grant, and no arrangement considered here enlarges it.

Each investor holds an *account*, and the account keeps a *record* r_i: the investor's own actions, such as a signature or a report, the positions held in the account, whether taken by him or by a manager acting for him, and the outcomes of those positions. Outcomes depend on the account's positions and on the state of nature, not on what other accounts hold; an investor's background wealth, liabilities and existing commitments belong to his account, so that a complete record contains everything about the state that bears on its owner's terminal wealth.

Payoffs are certainty equivalents in dollars, and transfers are sure payments. Under constant absolute risk aversion,

$$\mathrm{CE}_i(X) = -\frac{1}{\alpha_i} \log \mathbb{E}\left[e^{-\alpha_i X}\right], \qquad \alpha_i > 0, \tag{2}$$

a sure transfer t satisfies $\mathrm{CE}_i(X + t) = \mathrm{CE}_i(X) + t$, so certainty-equivalent dollars are a common numeraire, with risk neutrality as the limiting case. The aggregate welfare measure and every statement about transfers rely on this property; the frontier-equivalence and compression theorems do not. A sum of certainty equivalents is never read as a gain to a particular investor, and a transfer counts as available only when it is admissible under the investor's mandate, financeable when due and enforceable. A risky payment is not a sure transfer, so that where a result turns on lotteries over whether an investment forms, it is stated for risk-neutral investors; for the blind-pool result of Section 3 its counterpart under constant absolute risk aversion is given beside it, and

for the screening results in Appendix B.1, while the results of Section 4.4 on forming a book are established for risk-neutral investors only.

Investor i's *outside value* o_i is the certainty equivalent of the best portfolio available to him in his own account without the shared program, net of the cost of implementing it there. Existing wealth, liabilities, unfunded commitments and liquidity requirements enter that portfolio before a new investment is evaluated. Anything that improves what a separate account can do, such as a lower fee, broader access or cheaper customization, raises o_i.

2.2 Three kinds of term

The terms of an investor's contract are payments, obligations and permissions, and they are distinguished by what they may depend on.

Definition 1 (Own, escrowed and joint terms)**.** A term of investor i's contract is *own* if, given the state of nature and the actions taken by him or on his behalf in his account, it does not vary with the actions of other investors; *escrowed* if it varies with them only through whether the commitments received form the investment; and *joint* otherwise.

The definition turns on invariance rather than on what a term mentions. A payment to an investor whose account committed but holds no position is escrowed, because whether it holds a position depends on whether others committed, and a fee on the outcome of a position that a manager chose for the account is own, because given that position the outcome depends only on nature.

A separately managed account writes own terms: a management fee on its assets, a performance fee on its returns, a list of positions its manager may take. An escrow agent who holds subscriptions until a minimum is reached and returns them otherwise writes escrowed terms, the simplest terms that depend on other investors, since they do so only through a single event. A fund writes joint terms as well, such as a mandate that permits one combination of positions across investors and excludes another, a promise to pay subscribers if a closing fails, or compensation computed on the records of several accounts; nothing in the definition requires pooled ownership, and such terms can be written over separately owned accounts by any arrangement that observes their records and can enforce the terms, whose cost Section 5 counts among the costs of records.

A deliberately restrictive benchmark fixes what own terms alone achieve. Suppose that every claim an arrangement could allocate is available to each investor separately, on the same terms and in any quantity; that no claim requires a formation cost, a minimum scale or a minimum ticket; that no capacity is scarce; and that each investor's payoff depends only on his own holdings.

Proposition 1 (Rectangles)**.** *Under these conditions every allocation attainable under joint terms is attainable under own terms, and the set of attainable joint allocations is the product of the sets each investor can attain alone.*

Each investor's attainable set does not depend on what others choose, so any holding that a joint plan assigns to an investor he can obtain alone on the same terms. A strictly positive

gain from terms that depend on other investors therefore requires at least one of four sources: a missing market, such as the market for insurance against one's own need for liquidity, which a pooled fund supplies by making each investor's redemption terms depend on others' (Diamond and Dybvig, 1983; Jacklin, 1987); a formation cost or minimum scale or ticket; scarce capacity; or a dependence of one investor's payoff on another's actions, of which delegation to a common manager is the leading instance. In liquid markets for broad risks, mixtures of a few common funds come close to every investor's optimum, and pooling is a question of cost. Sections 3 and 4 ask what minimum scale adds, Section 5.1 what delegation adds, and Appendix B.14 returns to scarce capacity as the condition under which the compression theorem fails; liquidity insurance, whose dependence on other investors is plain, is left aside. The benchmark is not a description of general equilibrium, in which aggregate resources and market clearing constrain what investors can hold together; it isolates the assumptions under which terms that depend on other investors have no role, so that each departure from it can be priced.

2.3 What a fund holds in common

Write $x = (x_{ij})$ for the matrix of holdings, investor i's exposure to opportunity j being the entry in row i and column j. A pooled fund restricts the rows of x to be proportional to one another and gives its manager *discretion* over the columns, placing *principal* whose permissions allow it to finance what the investments require, while a separate account restricts nothing but can commit only on its own terms. A pooled fund also binds its investors before the investments and their terms are known. Section 3 prices the support that commitments need; Section 4 prices compatible principal and discretion and, through the capacity a closing needs, the moment at which participation binds; and Section 5 prices what individual exposure and individual terms cost instead.

2.4 The formation game

A shared program requires participation by several investors. Investor i chooses an action a_i from a set A_i that contains the null action 0, remaining outside. A profile $a = (a_1, \dots, a_n) \in A = \prod_i A_i$ either forms the program or fails, and $\mathcal{F} \subseteq A$ denotes the formed profiles. Investor i's payoff is

$$u_i(a; o) = o_i + g_i(a; o), \tag{3}$$

where g_i is his incremental payoff from participation, net of his share of formation and implementation costs and of his expense of standing ready. Indexing g_i by o records that a technology which moves outside values may also move incremental payoffs. Two assumptions define the baseline game.

Assumption 1 (Independent outside options)**.** $g_i(0, a_{-i}; o) = 0$ for every i and every a_{-i}.

Assumption 2 (Minimum scale without failure rewards)**.** No profile in which a single investor takes a non-null action is formed, and $g_i(a; o) \leq 0$ for every i whenever $a \notin \mathcal{F}$.

Assumption 1 says that an investor who stays out obtains his outside value whatever others do. Assumption 2 says that no investor can form the program alone and that a failed attempt returns capital or consumes costs but never pays; it describes a program whose only escrowed term is the conditional subscription itself. Section 3 relaxes the assumption deliberately, admitting payments upon failure, which are the instrument that assurance uses, and, for comparison, investors able to form the investment alone.

The game describes a decision to undertake a new commitment, and the date of that decision fixes what an action means. Remaining outside means declining a proposed commitment, not withdrawing from an accepted one; existing investments, unfunded commitments to other vehicles and liquidity requirements enter the outside value, while capital already bound to a program belongs to no action set, and where some participants are already bound the game applies to the commitments that remain open, as in Appendix B.9. The comparative statics that follow therefore apply to new programs, to the next fund in a series, to incremental commitments and to amendments that require fresh consent, and not to a closed-end vehicle that unravels when an investor's alternatives improve, except where leaving is itself a contractual right, as with the redemption rights of Section 7.3. Lock-in moves the participation decision earlier without removing it, and an investor deciding whether to sign values the whole commitment, including its duration, its capital calls and the alternatives it forecloses (Gourier, Phalippou and Westerfield, 2024).

Let $\mathcal{E}(o)$ be the set of pure Nash equilibria and $W(a; o) = \sum_i u_i(a; o)$ aggregate welfare, which is assumed bounded above on A so that every supremum below is finite. The extreme equilibrium welfare levels and the *selection gap* are

$$L(o) = \min_{a \in \mathcal{E}(o)} W(a; o), \qquad H(o) = \sup_{a \in \mathcal{E}(o)} W(a; o), \qquad G(o) = H(o) - L(o). \tag{4}$$

No rule for selecting among equilibria is imposed.

Lemma 2 (The outside floor)**.** *Under Assumptions 1 and 2, the profile in which every investor remains outside is a pure equilibrium, every pure equilibrium gives each investor at least his outside value, and* $L(o) = \sum_i o_i$.

Proof. If all others remain outside, a non-null action fails to form the program and yields $g_i \leq 0$, so remaining outside is a best response. In any equilibrium, investor i can remain outside and obtain o_i, so $g_i(a; o) \geq 0$. The all-outside profile attains $\sum_i o_i$. □

2.5 Three sets and the value of coordination

A profile a is *individually acceptable* at o if $g_i(a;o) \geq 0$ for every investor; $\mathcal{A}(o)$ denotes the set of such profiles, which contains the all-outside profile. For a set Y of profiles write $g(Y;o) = \{g(a;o) : a \in Y\} \subseteq \mathbb{R}^n$. Lemma 2 gives the nesting

$$g(\mathcal{E}(o);o) \;\subseteq\; g(\mathcal{A}(o);o) \;\subseteq\; g(A;o). \tag{5}$$

The three sets describe what voluntary choices support, what each investor would accept and what the technology permits under the program's terms. Their upper boundaries are the *equilibrium*, *acceptable* and *feasible frontiers* of the investment problem. The investment frontier, in the sense of this paper, is the equilibrium frontier: the boundary of payoffs that investors actually attain, whose distance from the other two is a matter of organization.

Contracts add a fourth set. A contract c from an admissible class C specifies transfers, conditions and contingent payments, and it modifies the baseline game. Let $\mathcal{E}_c(o)$ be the pure equilibria of the modified game, $W(a;c,o)$ its aggregate welfare net of the contract's real costs, and $\mathcal{I}(o;C)$ the set of payoff vectors attained at some $a \in \mathcal{E}_c(o)$ for some $c \in C$. If C contains the null contract, equilibrium payoffs of the baseline game belong to $\mathcal{I}$.

Aggregate surplus gives a scalar summary. Let $V(o) = \sup_{a\in A} W(a;o)$ be the largest aggregate certainty equivalent attainable under the program's terms, whether or not every investor accepts it, and let $W^*(o) = \sup_{a\in\mathcal{A}(o)} W(a;o)$ be the largest among individually acceptable profiles. By Lemma 2, $L \leq H \leq W^* \leq V$, and the value of coordination decomposes as

$$V - L = \underbrace{(H - L)}_{\text{selection}} + \underbrace{(W^* - H)}_{\text{instability}} + \underbrace{(V - W^*)}_{\text{participation conflict}} . \tag{6}$$

The selection gap $G = H - L$ is surplus that equilibrium selection could recover. The instability term is surplus from allocations that every investor accepts but that no equilibrium supports. The participation-conflict term is surplus that requires some investor to accept less than his outside value; the paper calls it the *distribution gap*, $\Lambda = V - W^*$, because for a fixed set of participants only a change in what investors receive can recover it. The surplus that no equilibrium of the baseline game delivers, $V - H$, is the sum of the instability term and the distribution gap. The completion property of Section 6 eliminates the instability term, so that under it $V - H = \Lambda$, and every unit of surplus beyond the best equilibrium requires new terms.

2.6 The standard of implementation

Support in one equilibrium differs from a guarantee in every equilibrium. A contract can make an attractive allocation an equilibrium while leaving the all-outside equilibrium intact, and a designer concerned with the worst case evaluates

$$\sup_{c \in C} \inf_{a \in \mathcal{E}_c(o)} W(a; c, o) \tag{7}$$

over contracts with nonempty equilibrium sets, which can lie strictly below the largest welfare in $\mathcal{I}(o; C)$. A contract *implements* an outcome if the outcome obtains in every equilibrium, Bayesian where types are private, including equilibria in weakly dominated strategies. The standard reflects the fact that committing to an investment is never free: any part of the expense of standing ready that a contract leaves uncovered turns a weakly dominated refusal into a strict best response to refusal, and an equilibrium that a small residual expense would make strict is one that an investment cannot rely on avoiding. Where a choice carries no such expense, as a report to a mechanism does, weak dominance is the appropriate standard, and where commitments are made in public rounds, subgame perfection replaces it, or perfect Bayesian equilibrium where willingness is private. Section 3.2 strengthens the standard where it compares contracts of every form contingent on investors' commitments, requiring the outcome to be the only one consistent with rationalizable play, and for closed cohorts and counts it also considers the weaker requirement that the outcome be the unique correlated equilibrium. Section 3.4, where commitments are made in rounds, uses backward iterated dominance, which expresses common belief in future rationality (Perea, 2014). Where willingness is private and commitments are made in public rounds, a perfect Bayesian equilibrium is understood as in Fudenberg and Tirole (1991) for games with observed actions and independent types: each investor's belief about the willingness of investors who have not yet moved equals the prior at every history, on or off the path of play. The assignment mechanisms of Section 4 are evaluated by whether truthful reporting is a dominant strategy, a standard that the uniform-price assignment meets exactly when its premium is retained and participation is settled before reports are made, and approximately when the premium is returned within the book, while under a retained veto the investor who sets the price can gain by shading his report, and the expected-externality mechanism of Proposition 25 is evaluated by Bayesian equilibrium; decisions to accept a place or to enroll in a book are evaluated by weak dominance where it applies and by equilibrium where investors know their own types when they enroll (Section 4.4).

3 Commitment support

Committing together is a formation problem, and when commitments are simultaneous the outcome in which nobody commits is an equilibrium whatever investors would gain, because a commitment that others do not match is wasted. This section establishes what removing that outcome requires: terms written account by account remove it only by transfers on the equilibrium path that someone

must finance, payments conditioned on failure remove it with nothing paid in equilibrium, and among all terms that depend on who else commits, however elaborate, none removes it robustly with a promise at or below the bound that the escrow approaches (Theorems 3 to 5). The promise is reusable because it is never drawn when willingness is known (Proposition 8), a calendar divides it among investors who believe in one another's future rationality (Theorem 11), and private willingness requires it to be paired with a charge or placed within an order (Theorem 13). What collective commitment requires is therefore support, either transfers on the equilibrium path or contingent capacity, the ability to pay in a state in which the investment does not exist, and the cheapest robust form of the support is the second. Results that compare pure profiles of commitments involve only sure outcomes and hold for any risk preferences, and the bound under the uniqueness of correlated equilibrium holds for strictly increasing concave utilities when the wealth an investor has on declining is independent of all actions and of the correlating device; results that turn on lotteries over formation, the screening and blind-pool results below and the waste result of Appendix B.1, are stated for risk-neutral investors, with the conditions for investors with constant absolute risk aversion given beside the blind-pool result and, for screening, in that appendix.

3.1 Own payments and escrowed payments

Let a *formation structure* be a nonempty family $\mathcal{F}$ of sets of investors that excludes the empty set and is closed under enlargement: the investment forms when the investors who commit constitute a set in $\mathcal{F}$. With binary commitments a profile of actions is identified with the set of investors who commit, and the formed profiles of Section 2 with a formation structure. A count of seats, $\{S : |S| \geq \nu\}$, a threshold of capital, $\{S : \sum_{i\in S} q_i \geq K\}$, and a list of required roles are all formation structures, and a *closed cohort* is the structure $\{S : S \supseteq F_0\}$ in which the investment forms exactly when every member of a given coalition F_0 commits. Until Section 3.5 every investor's willingness is common knowledge, and each committed investor i bears an expense of standing ready, $h_i \geq 0$, which does not depend on who else commits; if the committed set S belongs to $\mathcal{F}$, he obtains a gain $g_i(S) > 0$ over his outside value net of that expense, and if it does not he loses h_i, while an investor who declines obtains his outside value. Two kinds of payment can be added. An *own* payment $t_i \geq 0$ goes to investor i if he commits, whatever others do; nothing is paid for refusing, since an investor owes nothing for declining an offer. An *escrowed* payment $\beta_i \geq 0$ goes to investor i if he commits and the investment fails, and the largest claim it can create, the *liability* of the schedule, is

$$R(\beta) = \max_{S\notin\mathcal{F}} \sum_{i\in S} \beta_i. \tag{8}$$

Theorem 3 (Escrow or anchor)**.** *Let commitments be simultaneous and let* $R^* = \min_{F\in\mathcal{F}} \left(\sum_{i\in F} h_i - \max_{i\in F} h_i \right)$.

(i) A schedule of own payments implements formation if and only if the set $S^+ = \{i : t_i > h_i\}$ *belongs to* $\mathcal{F}$ *or does so after adding one investor. The payments that such a schedule*

makes in equilibrium to the investors in S^+ are at least R^*, and strictly more unless a single investor can form the investment alone; R^* is their infimum.

(ii) A schedule of escrowed payments implements formation if and only if the set $B^+ = \{i : \beta_i > h_i\}$ belongs to $\mathcal{F}$ or does so after adding one investor. Such a schedule pays nothing in equilibrium, and the infimum of its liability is R^*.

(iii) Either infimum is attained, at zero, if and only if a single investor can form the investment alone.

Part (ii) is the companion paper's assurance theorem (Komkov, 2026b), restated for comparison with part (i). The construction behind both parts is a chain of strict dominance. Take a coalition F that attains R^*, leave one of its most expensive members unassured, and pay every other member slightly more than his expense. Those members commit whatever anyone else does, because committing pays strictly whether or not the investment forms; the remaining member then completes F by committing, whatever he believes, because his commitment forms the investment and membership pays him strictly; and every other investor joins, because the investment forms whether or not he does. Conversely, any schedule that implements formation strictly assures all but at most one member of some forming coalition, for otherwise two unassured investors could refuse together and leave the others unable to form it, and the expenses of those assured members are at least R^*. Neither direction involves a belief or an attitude to risk, and sufficiency uses only the signs of the members' gains from formation. The two kinds of payment also combine: an investor paid t_i on commitment and β_i on failure commits whatever others do exactly when $t_i + \beta_i > h_i$, and both characterizations hold with that set in place of S^+ and B^+.

The two parts differ chiefly in what they ask of the party that makes the payments. Own payments are transfers in the state that occurs, more than R^* in total to the investors they assure, and someone must make them: an outsider, or a member whose own gain from formation covers them, as when the one unassured member of a forming coalition pays the others on commitment, which implements formation with no net subsidy whenever his gain exceeds the payments, although his obligation to pay does not depend on whether the investment forms, so that he too must be able to pay in the state in which it fails. Escrowed payments are transfers in a state that does not occur, so that nothing changes hands in equilibrium, and the promisor needs only the capacity to pay the liability in the failure state, whose cost over the closing window is $\iota\bar{\tau}R^*$ at an annual cost ι of holding capacity over a window of length $\bar{\tau}$. On the welfare measure of Section 2, which nets transfers, the own scheme costs nothing and the escrow costs the holding of its capacity; the escrow economizes the promisor's cash and the participation of whoever would otherwise have to pay, and it requires a promisor able to pay in a state in which the investment does not exist.

Part (iii) describes the alternative, which rests on the fact that an investor who can form the investment alone needs no assurance; an underwriter who commits to take an entire issue makes himself such an investor, and the paper calls such an investor an *anchor*, in a narrower sense than the market's, for which an anchor is any large early investor; such an investor plays the part of the first mover in an order when his commitment binds (Section 3.4). An anchor bears the risk

and illiquidity of holding the whole issue until it can be resold, together with the cost of reselling it, since the capital itself buys an asset worth about what it costs. Measured in the same units, the two routes differ by orders of magnitude: in the economy of Appendix B.3.3, where four indispensable investors with expenses of standing ready of \$500, \$800, \$600 and \$1,000 finance a \$400,000 purchase, an anchor must be able to take the whole purchase onto his balance sheet and carry its risk for as long as resale takes, whereas an escrow must be able to pay \$1,900, about half of one percent as much, and only in a state that does not occur, so that it bears no risk of the investment at all. The law of contingent offerings supplies a refund rather than an escrow: an offering sold on an all-or-none or minimum basis must hold subscribers' money in escrow until the contingency is met and return it if it is not (Securities and Exchange Commission, n.d.a,n), which makes the conditional subscription credible but pays nothing above subscribers' expenses, so that Theorem 3 prices what must be added to remove the equilibrium in which nobody subscribes; Section 7 returns to anchors and refunds in practice.

3.2 The cheapest joint term

Own and escrowed payments are the two simplest classes, and a fund can write far more elaborate terms: a payment to an investor that depends on exactly which others committed, a bonus to whoever commits first among a named group, a payment to an investor who declines when certain others have committed. A *payment scheme* is any such rule in the commitment game. It specifies, for every committed set S that fails, a nonnegative payment $b_i(S)$ to each investor, committed or not, and for every set that forms, transfers that leave each member's net gain from formation positive and nonnegative payments to investors who declined. Its liability is the largest total it can owe when the investment fails,

$$R(b) = \max_{S \notin \mathcal{F}} \sum_{i \in N} b_i(S), \tag{9}$$

which reduces to (8) for an escrowed schedule, $b_i(S) = \beta_i \mathbf{1}\{i \in S\}$. A scheme implements formation *robustly* if formation is the unique rationalizable outcome of the commitment game it induces (Bernheim, 1984; Pearce, 1984): every profile of actions that survives the iterated removal of actions that are never best responses forms the investment. With two actions an action is never a best response, whether beliefs are independent or correlated, exactly when the other strictly dominates it, so robust implementation is implementation by iterated strict dominance, and it implies implementation in every Nash equilibrium, mixed equilibria included.

Theorem 4 (Escrow optimality)**.** *Let commitments be simultaneous and willingness known, and suppose no single investor can form the investment alone. Every payment scheme that implements formation robustly has liability greater than R^*. Escrowed schedules implement formation robustly with liabilities arbitrarily close to R^* and pay nothing in equilibrium.*

The proof follows the order in which iterated dominance forces investors to commit. Robust implementation requires that someone's refusal be removed first, and it can be removed only if

committing pays him strictly even when everyone else refuses; the removals continue until the investors forced to commit form the investment. At the last failing set before that happens, every investor already forced to commit had to be paid more than his expense, since otherwise the profile in which exactly those investors commit would have left his refusal standing, and that failing set together with the last investor forced to commit forms the investment. The scheme therefore owes, at a single failing profile, more than the expenses of a forming coalition less one of its members, which is at least R^*. Nothing in the argument depends on what the scheme pays when the investment forms, on how payments vary with the identities of other committers, or on whether investors who decline are paid, and so the generality of a scheme buys nothing that an escrow does not already supply. Stated as a matter of feasibility, the theorem says that a promisor whose resources in the state of failure are W can implement formation robustly if and only if $W > R^*$: it concerns the capacity a promisor must have, not welfare. It also gives a complete-information counterpart to the question that Sákovics and Steiner (2012) ask of global games, where strategic uncertainty is the source of failure, which agents a planner should subsidize to secure coordination; here the answer is every member of a forming coalition that attains R^* except its most expensive.

Rationalizability is a demanding standard, and the bound extends in part to a weaker one. A scheme makes formation the unique *correlated* equilibrium if every correlated equilibrium of the commitment game it induces places probability one on profiles that form the investment (Aumann, 1974). Every Nash equilibrium is a correlated equilibrium, and every correlated equilibrium is supported on rationalizable profiles, so the requirement is weaker than robust implementation and stronger than implementation in every Nash equilibrium; by Aumann (1987), it says that formation obtains in every Bayesian equilibrium of every game obtained by letting investors observe signals that do not affect their payoffs, so that no sunspot or mediator, however it is chosen, can coordinate investors on failure, and by Kajii and Morris (1997) a unique correlated equilibrium is robust to incomplete information, in the sense that every game of incomplete information in which the known payoffs obtain with high probability has an equilibrium whose distribution of actions is close to it. The requirement concerns what an arrangement can guarantee, not what a helpful coordinator can achieve: formation is already an equilibrium of the game without payments, and a mediator who recommended it to investors willing to follow would need no capacity at all, but nothing in the game makes investors follow him rather than a recommendation to refuse.

Theorem 5 (Correlated robustness)**.** *Suppose no single investor can form the investment alone, let $\nu_{\min}$ be the number of members of the smallest coalition in $\mathcal{F}$, and let R_c be the sum of the $\nu_{\min} - 1$ smallest expenses of standing ready. Every payment scheme that pays nothing to investors who decline and makes formation the unique correlated equilibrium has liability greater than R_c, whether investors are risk neutral or have strictly increasing concave utilities of their wealth, provided that each member's gain from formation is positive in expected utility, that the wealth an investor has when he declines is independent of all investors' actions and of the correlating device, and that his expected utilities and his expected marginal utility at that wealth are finite. If the structure counts the commitments of a set M of members, $\mathcal{F} = \{S : |S \cap M| \geq \nu\}$ for some $\nu \geq 2$,*

as a closed cohort ($M = F_0$ and $\nu = |F_0|$) and a count of seats ($M = N$) do, every such scheme has liability greater than R^*, which is then the sum of the $\nu - 1$ smallest expenses among the members.

The proof is by linear duality, and its central step is short enough to describe. Because an investor who declines is paid nothing, every investor strictly prefers committing at the profile at which everyone commits, and formation is then the unique correlated equilibrium exactly when there are nonnegative weights, one for each investor, such that at every failing profile the weighted gains of the investors who would enter it, each entering alone, are positive. At a failing profile from which no single entry forms the investment, those gains are payments above expenses at the failing profiles one entry larger. Weighting each such profile by the product of the weights of the investors committed at it gives every committed investor at the larger profile the same weight, namely the product of the weights of all who are committed there, so that summing turns the conditions into one inequality: a weighted average of the total payments at failing profiles exceeds the same average of the expenses of the investors committed at them. For a count, the profiles used are those at which every outsider and $\nu - 2$ members have committed, the expenses of the $\nu - 1$ committed members at a profile one larger are at least R^*, and the liability, the largest payment owed at any failing profile, is at least the average; for a closed cohort the weight of each such profile is inversely proportional to the weight of the member missing from it. In general the profiles used are those at which $\nu_{\min} - 2$ investors have committed, and the bound is R_c, which equals R^* for a count of seats among all investors and can be smaller otherwise. The escrow attains R^* because it makes formation the unique rationalizable outcome, so for counts the least liability is R^* under every standard between iterated strict dominance and the uniqueness of correlated equilibrium. Risk aversion leaves the bound intact because an investor's gain in expected utility from committing at a failing profile is at most his expected marginal utility at the wealth he has when he declines times the payment net of his expense, so that a certificate in utility yields one in money with weights scaled by expected marginal utilities.

The second statement does not extend to every structure, and the first is the most that holds in general. Let investors 1 and 2 be required and either of investors 3 and 4 suffice, with expenses of 4, 4, 3 and 3, so that $R^* = 7$ and $R_c = 6$, let investors be risk neutral, and let every member's gain at a formed profile be 20. Pay 4.5 to investor 1 or 2 and 3.5 to investor 3 or 4 if he alone commits; 3.25 to each of two committed investors of the same kind; 3.4 to the required investor and 3.1 to the other in a committed pair of different kinds; and 4.1 to the required investor when he and both others commit. The liability is 6.5, below R^*, and formation is the unique correlated equilibrium, although no action is dominated, so that failure remains rationalizable as Theorem 4 requires. The scheme works only because the gains are large and investors are risk neutral: with gains of 14 rather than 20, or with gains of 20 and constant absolute risk aversion of 0.05, it admits a correlated equilibrium in which the investment never forms, and as gains grow the least liability falls toward R_c, a liability of about 6.04 sufficing when they are 200. Weakening the standard from rationalizability to the uniqueness of correlated equilibrium can therefore lower the requirement below R^* only for structures that are not counts, in which investors play different roles, never below R_c, and, as the example shows, by exploiting what investors gain, which ties the capacity to

the fees they are charged; the robust standard rules out that dependence. The restriction to schemes that pay nothing to investors who decline is needed as well: in a count of three seats among four investors whose expenses are 1, so that $R^* = 2$, a payment of one hundredth to an investor who declines while the other three form, equal to his gain from joining them, leaves him indifferent there and lets a scheme with liability 1.1 make formation the unique correlated equilibrium, as the proof of Theorem 5 in Appendix A shows.

The two theorems price, over all profile-contingent schemes, a promise that the literature on contracting with externalities has noted and priced only in particular forms. Segal (2003) and Bernstein and Winter (2012) exclude payments that depend on other agents' choices, Winter (2004) ties bonuses to success alone, and Halac, Kremer and Winter (2020), who rule out positive payments upon failure, observe that a firm whose contracts could condition on third parties' choices would extract the full surplus of the investors it finances, and in an extension let the firm's own capital insure investors against failure, a capacity-constrained form of the escrow. Theorems 4 and 5 identify what such conditioning requires: the capacity to pay R^* in a state in which the investment does not exist. Under simultaneous commitments that capacity cannot be dispensed with, and it is also sufficient for extracting the surplus.

Corollary 6 (Contingent capacity)**.** *An escrowed schedule that implements formation robustly continues to do so when each member is charged, upon formation, any fee smaller than his gain. A promisor able to pay slightly more than R^* when the investment fails can therefore collect nearly the whole surplus of formation while paying nothing in equilibrium, and the capacity required for robust implementation does not depend on the investors' gains, their beliefs, their attitudes to risk or the fees they are charged.*

The case for robustness is that it guarantees formation whatever investors believe about one another, which is what a promisor who must commit capacity before those beliefs are known requires; cheaper schemes exist, but they work only for investors who coordinate on them. In a game with three risk-neutral, indispensable investors whose expenses of standing ready are 1, 2 and 2 and whose gains from formation are 3 each, $R^* = 3$, yet a scheme that pays the first investor 2 if he alone commits and the second 2.25 if exactly the first two commit, and nothing otherwise, has liability 2.25 and formation as its only Nash equilibrium, mixed equilibria included. It works by exploiting the investors' gains, which make every failing profile unstable, but no action is dominated under it, so refusal remains rationalizable for every investor, and it admits a correlated equilibrium in which the investment fails with probability 0.9975.

Proposition 7 (Fragility of cheaper schemes)**.** *Under any payment scheme that does not implement formation robustly, some investor's refusal is rationalizable. If payoffs are rich, in the sense that for each investor and each action some payoff type makes that action strictly dominant, then that investor's refusal is his unique rationalizable action at types arbitrarily close, in the product topology on hierarchies of beliefs, to common knowledge of the scheme and the game.*

The second statement is the structure theorem of Weinstein and Yildiz (2007), applied to the commitment game. A scheme that relies on coordination therefore cannot be distinguished, by

any finite amount of evidence about investors' beliefs, from one under which an investor refuses for certain, and it is not continuously implementable in the sense of Oury and Tercieux (2012). The structure theorem cuts both ways, since formation without any payment is also the unique rationalizable outcome at nearby types for which some investor's commitment is dominant; what it establishes is that a prediction of formation under a scheme that is not robust rests on beliefs that no finite evidence can confirm, so that robustness is the standard of a designer who wants a guarantee rather than a prediction. The gap between the two standards is an instance of the general difference between Nash and rationalizable implementation (Bergemann, Morris and Tercieux, 2011); in the supermodular games of the divide-and-conquer literature the two coincide (Halac, Kremer and Winter, 2020; Halac, Lipnowski and Rappoport, 2022), and the gap opens here because terms contingent on others' commitments can undo the complementarity of commitments that makes a failing profile self-confirming. Robustness has a price at the margin as well: the escrow's members commit by strict dominance only if their payments exceed their expenses, and a margin of ε per assured member protects formation against perturbations of their payoffs smaller than ε, so that robustness against perturbations of size ε costs liability R^* plus ε for each assured member; the unassured member and the investors outside the coalition are protected only to the extent of their net gains from formation, which the fees of Corollary 6 reduce. Extraction meets a second limit when investors can make side payments: if the margins by which assured members are paid above their expenses exceed, in total, the net gains left to the members at formation, the members can share the escrow's payment by agreeing to fail, so that a promisor who extracts nearly the whole surplus must keep the margins smaller still. A refund that reimburses expenses exactly, $\beta_i = h_i$, is the escrow with no margin at all: it leaves an investor indifferent between committing and refusing when others refuse, so refusal remains an equilibrium, if one in weakly dominated strategies, and any positive margin removes it.

3.3 A portfolio of promises

With willingness known, contingent capacity is never drawn when formation is implemented, and this makes it reusable. Let a single promisor back several investments $k = 1, \ldots, K$, each with its own investors, formation structure, expenses and least liability R_k^*, and let it honor escrowed claims out of a single capacity C in a fixed order of seniority among the investments: a claim arising from investment k is paid in full if the capacity left after the claims of more senior failed investments allows. The investments have disjoint sets of investors, payments at formed profiles are not drawn from C, investors in all the investments commit simultaneously, and the structures, schedules and order of seniority of all of them are common knowledge, since an investor's assurance depends on what the investments senior to his own can claim.

Proposition 8 (Portfolio of promises)**.** *If C exceeds $\max_k R_k^*$, there are escrowed schedules under which the formation of every investment is the unique rationalizable outcome of the joint commitment game, and nothing is paid. Conversely, if formation of every investment is the unique rationalizable outcome under any payment schemes honored out of a common capacity C, then*

$C > \max_k R_k^*$ *whenever no investment can be formed by a single investor, whereas separate promisors would need* $\sum_k R_k^*$ *between them.*

Seniority makes the argument a chain of dominance across investments. The investors of the most senior investment are assured whatever happens elsewhere, so it forms by the argument of Theorem 3; once it forms, it makes no claim on the capacity, which then assures the investors of the next, and so on down the order. The capacity is a public good among the investments the promisor backs, and its cost per investment falls as their number rises, in inverse proportion to it when their requirements are equal. It is an economy of scope that requires a common promisor but not common ownership: investments backed by separate promisors cannot reproduce it, while separately owned accounts backed by one promisor can, so that the part of a fund's work that only shared terms can do is that of a promisor whose contingent capacity serves every investment it backs. The argument parallels the diversification that makes delegated monitoring cheaper for an intermediary than for its depositors (Diamond, 1984), with the difference that here nothing is diversified, since every claim is avoided rather than averaged, and it complements the agency explanation of ex ante pooling in Axelson, Strömberg and Weisbach (2009), in which a fund raised before its deals disciplines a manager who would otherwise invest in bad ones. The economy has a limit that the proposition makes explicit: it rests on the capacity never being drawn. When willingness is private, the screening escrow of Theorem 13(ii) is paid whenever some investor proves unwilling, so that claims arise on the path of play, and capacity must then cover the claims that can arise together, an ordinary problem of insurance in which pooling lowers the capacity per investment as independent claims average out but does not reduce it to the largest single requirement.

3.4 Order and time

Simultaneity is essential to Theorems 3 to 5. If investors commit one at a time in a publicly observed order, each commitment binding if the investment forms, the last investor whose commitment can complete a forming coalition commits, and backward induction carries the argument to the first. What time is worth therefore depends on the standard adopted for dynamic play. Let commitments be made in ℓ public rounds, each investor committing in a round that the arrangement assigns to him, commitments within a round being simultaneous and observed before the next, and each binding if the investment forms; write N_r for the investors assigned to round r. A scheme implements formation by *backward iterated dominance* if formation is the only outcome that survives the following procedure: at every history, remove any action of an investor moving there that is strictly dominated given every combination of the remaining actions of the others moving there and of the remaining behavior at every later history, and repeat until nothing more is removed. The procedure is the backward dominance procedure of Perea (2014), which characterizes common belief in future rationality, and it is close to the backward rationalizability of Penta (2012); with one round it is iterated strict dominance, and implementation by it implies implementation in every subgame-perfect equilibrium.

Proposition 9 (Order)**.** *With known willingness and commitments made one at a time in a publicly observed order, each binding if the investment forms, formation is implemented without any payment, by backward iterated dominance and in every subgame-perfect equilibrium.*

A lead investor whose commitment binds first and publicly, and a seed commitment that binds before others are solicited, are orders in this sense; a lead whose announced terms do not bind is not, and Section 7 shows that venture term sheets are of this kind. The proposition is the formation-game form of results known for threshold public goods, in which an announced lead gift removes the equilibrium in which nobody gives (Andreoni, 1998) and irreversible sequential contributions coordinate on provision (Admati and Perry, 1991; Marx and Matthews, 2000; Gale, 2001), and Halac, Kremer and Winter (2020) observe that sequential offers secure investment as the unique subgame-perfect outcome while doubting that investors would rely on others who have yet to move; Varian (1994) shows that sequencing can lower provision when contributions are continuous, so that the advantage of an order depends on the threshold. An order is a sequence of dates, however, and dates are scarce: a seller's exclusivity period, an auction's deadline or a closing date fixed by a lender admits only so many rounds in which commitments can be made, documented and observed before the next are made. With own and escrowed payments, an investor is *assured* if $t_i + \beta_i > h_i$, and for a coalition F, $h_F^{(\ell)}$ denotes the sum of the ℓ largest expenses among its members, or of all of them if it has at most ℓ members.

Proposition 10 (Rounds with time-invariant escrow)**.** *With known willingness and ℓ public rounds, a schedule of own and escrowed payments and an assignment of investors to rounds implement formation in every subgame-perfect equilibrium if and only if the assured investors, together with at most one other investor from each round, constitute a coalition in $\mathcal{F}$, and such a schedule also implements formation by backward iterated dominance. The infimum of the liability of an escrowed schedule that does so, and of the payments that an own schedule that does so makes in equilibrium, is*

$$R_\ell^* = \min_{F \in \mathcal{F}} \Big(\sum_{i \in F} h_i - h_F^{(\ell)} \Big), \tag{10}$$

which equals R^* when $\ell = 1$, falls weakly with each additional round, and is attained, at zero, if and only if some coalition able to form the investment has at most ℓ members.

Each round lets one more member of a forming coalition go unassured. In the economy of Appendix B.3.3, where all four investors are indispensable, the least liability falls from \$1,900 with a single round to \$1,100 with two, \$500 with three and nothing with four. A time-invariant escrow, however, keeps every assured investor insured until the last round, although an investor who committed in an early round needs protection only until his round closes: once it has, the commitments still to come are secured by the rounds that follow, and the insurance of the early investors protects them against an event that their belief in the others' future rationality has already

excluded. Releasing each round's escrow as the round closes exploits this. Given an assignment of investors to rounds, let

$$\hat{R}_\ell = \min_{F \in \mathcal{F}} \max_r \Big(\sum_{i \in F \cap N_r} h_i - \max_{i \in F \cap N_r} h_i \Big), \tag{11}$$

the least, over forming coalitions, of the largest round's expenses less its most expensive member, where rounds without members of the coalition are ignored. A *staged escrow* for a coalition F pays a committed member of F, other than a most expensive member of his round, slightly more than his expense if his round is the first in which the members of F did not all commit and the investment fails. It depends on others' commitments through the round in which commitments stall, and so it is a joint term in the sense of Definition 1, the simplest that time admits; with one round it is an escrow.

Theorem 11 (Staged escrow)**.** *Let willingness be known and commitments be made in ℓ public rounds under a given assignment. A staged escrow for a coalition that attains $\hat{R}_\ell$ implements formation by backward iterated dominance and in every subgame-perfect equilibrium, pays nothing in equilibrium, and has liability at most $\hat{R}_\ell$ plus an arbitrarily small amount. Every payment scheme contingent on the profile of commitments that implements formation by backward iterated dominance has liability at least $\hat{R}_\ell$, and greater unless $\hat{R}_\ell = 0$.*

The difference between the two kinds of escrow is large. With n indispensable investors of equal expense h, a time-invariant escrow needs $(n - \ell)h$ with ℓ rounds, while a staged escrow with rounds of nearly equal size needs $(\lceil n/\ell \rceil - 1)h$: with one hundred investors and ten rounds, $9h$ against $90h$, and against $99h$ for an escrow with a single round. Each round's escrow protects only the investors of that round, and only until the round closes, so that the calendar divides the promise as well as shortening it, and its least size falls roughly in inverse proportion to the number of rounds. With unequal expenses the best assignment solves a problem of scheduling, which balances the rounds' expenses net of their most expensive members; it is computationally hard in general, since with two rounds and two investors of very large expense, who are best placed in different rounds, it contains the partition problem, although the requirement of any given assignment is immediate. The theorem carries to a limited calendar the observation that transparency among agents lowers the cost of unique implementation under complementarity (Winter, 2006, 2010), a possibility that Segal (2003) excludes by ruling out sequential offers and that Möller (2007) and Genicot and Ray (2006) study for contracting with externalities, and it gives the saving in closed form.

Time is not free, and the comparison between a calendar and capacity can be made explicit. Let n indispensable investors have equal expenses h, let each round last τ, let every round after the first delay formation at a cost c, and let capacity cost ι per dollar per unit of time, each round's escrow being posted only while its round is open, as a promisor able to redeploy capacity between rounds can arrange; if the whole capacity had to be posted from the start, the carry would be $\iota\tau\ell \max_r(|N_r| - 1)h$ and the sizes of the rounds would matter.

Proposition 12 (Calendar or capacity)**.** *With ℓ nonempty rounds and a staged escrow, which implements formation by backward iterated dominance, the total cost of formation, the carry of the escrow plus the cost of delay, is $\iota\tau(n-\ell)h+(\ell-1)c$ plus an arbitrarily small amount, whatever the sizes of the rounds. Among calendars implemented by staged escrows, the cheapest is therefore a single round with an escrow if $c>\iota\tau h$ and a full order, one investor to a round, with no capacity at all if $c<\iota\tau h$.*

Because every investor is indispensable, a round that does not complete ends any prospect of formation, so the escrow of round r can be claimed only at its close and is released if the round completes; it is held only during round r, in the amount $(|N_r|-1)h$, and the carry summed over the rounds is $\iota\tau(n-\ell)h$. The total cost is linear in ℓ, which places the minimum at an end. The comparison takes the expense of standing ready to be the same however long the calendar, although an investor who commits in an early round stands ready for longer, which strengthens the case for capacity. A calendar is thus all or nothing: either delay is cheap enough that every investor should commit in turn, or it is costly enough that none should wait. The threshold is the carry of one investor's expense for one round, which is small, so the proposition predicts contingent capacity, or an anchor, wherever a deadline makes delay costly, and orders where the time that sequencing takes is consumed anyway.

What time buys depends also on the standard of rationality. Backward iterated dominance assumes that investors believe in one another's future rationality, and Penta (2012) shows that its predictions are robust to perturbations of beliefs that preserve the structure of the game, while Chen (2012) shows that the predictions robust to every perturbation of the game's normal form are those of normal-form rationalizability, under which an investor may refuse at a history he believes will not be reached. By that standard time can buy nothing even in the simplest case: with two investors who commit in turn and no payments, the second investor's plan to refuse after the first commits is a best response to the belief that the first refuses, which is a best response to that plan, so failure is rationalizable, and the staged escrow fails in the same way; whether time can buy anything in larger games under that standard is left open. The escrow placed before or alongside every other commitment survives every standard considered here, which is why the paper treats it, rather than time, as the support that collective commitment requires. The equilibrium in which nobody commits is also not coalition-proof when formation Pareto-dominates it (Bernheim, Peleg and Whinston, 1987), but a coordinated deviation from refusal presumes the communication and conditional commitment that an order or an escrow supplies.

3.5 Screening under private willingness

Private willingness creates a problem the preceding results do not face: a payment upon failure attracts investors who would rather the investment failed. Let all $n\geq 2$ investors be indispensable, $\mathcal{F}=\{N\}$, and let investor i be willing with probability $p_i\in(0,1)$, independently of the others, in which case formation gives him a positive gain, and unwilling otherwise, in which case formation costs him a positive amount, his loss from formation; gains and losses are drawn independently

of everything else, a committed investor bears $h_i > 0$ if the investment fails, and each investor knows only his own type. Formation is *willing* if the investment forms exactly when every investor is willing. Losses have *no gap* if, for every investor and every $\varepsilon > 0$, the probability that he is unwilling with a loss below ε is positive, which is the ordinary case when willingness turns on terms or valuations that vary continuously. A *charge* is a payment that an investor makes, either upon formation, a premium, or upon committing, a fee. Write $p_{-i} = \prod_{j \neq i} p_j$ for the probability that every investor other than i is willing; under a public order $1, \ldots, n$, write $p_{>k} = \prod_{j>k} p_j$ and $p_{\leq k} = \prod_{j \leq k} p_j$.

Theorem 13 (Screening with escrowed terms)**.** *(i) With simultaneous commitments and losses with no gap, no payment scheme that pays only investors who commit and imposes no charge makes willing formation the unique interim correlated rationalizable outcome.*

(ii) With simultaneous commitments and risk-neutral investors, suppose every willing type of investor i gains at least $\underline{g}_i > 0$ from formation. Fix an investor j. An escrowed schedule that pays every $i \neq j$ an amount $\beta_i > h_i$ upon failure and charges him a premium $f_i < \underline{g}_i$ upon formation, with

$$p_{-i} f_i > (1 - p_{-i})(\beta_i - h_i),$$

and that pays j an amount $\beta_j < h_j$ upon failure, with $(1-p_{-j})(h_j-\beta_j) < p_{-j}\,\underline{g}_j$, makes willing formation the unique interim correlated rationalizable outcome, whatever the distribution of losses; a fee upon committing that exceeds $(1 - p_{-i})(\beta_i - h_i)$ and is smaller than both $\beta_i - h_i$ and $\underline{g}_i$ serves in place of the premium.

(iii) In a public order with escrowed payments $\beta_k \leq h_k$ to the investors $k < n$ and no charge, with risk-neutral investors, every perfect Bayesian equilibrium yields willing formation if, for every $k < n$ and every willing type of investor k with gain g, $(1 - p_{>k})(h_k - \beta_k) < p_{>k}\, g$; with $\beta_k = h_k$ willing formation holds whatever the distributions of gains and losses and for any increasing utility. When every willing type of investor k gains g_k, the condition is also necessary, payments $\beta_k < h_k$ that satisfy it are claimed only by willing investors, only when an investor after them is unwilling, and the infimum of their expected cost is $\sum_{k<n} p_{\leq k}[(1 - p_{>k})h_k - p_{>k} g_k]^+$.

Part (i) holds because the first investor whose refusal iterated dominance removes must be paid more than his expense, on average over the profiles of other investors' willingness, at the failing profiles he might face, and an unwilling type of that investor whose loss is small enough then gains from committing, so willing formation is not the only rationalizable outcome; this is the problem that the companion paper, which holds the terms of formation fixed, solves with a gap between indifference and unwillingness or with reports (Komkov, 2026b). Part (ii) shows that the problem belongs to schemes that charge nothing for committing. A premium falls in the state that an unwilling investor wishes to avoid and an escrowed payment in the state he wishes for, so that a premium larger than the escrowed margin, scaled by the odds that everyone else is willing, makes committing unprofitable at every unwilling type however small its loss, while a willing investor

whose gain covers the premium commits whatever others do. The premium can be small, since the margin of the escrowed payment above the expense can be, but it requires willing investors' gains to be bounded away from zero: where they too accumulate at indifference, a fixed premium deters some willing types, and exact screening by simultaneous terms becomes a knife-edge. Part (iii) is the other route, and it needs no such bound: within an order an early investor bears only the risk that someone after him declines, and a payment equal to his expense, which no unwilling investor gains from collecting, makes that risk worth bearing for every willing one. Private willingness therefore adds no new kind of joint term to those of the preceding sections, since a premium is an escrowed term and an order is time.

Appendix B.1 extends the theorem to investors with constant absolute risk aversion, for whom the monetary inequalities are replaced by their exponential counterparts and a decision can reverse, and settles the order in which investors should commit when willingness is private. Under willing formation, with no investor committing after a refusal, the expected expense of standing ready wasted on failed attempts is $\sum_{k<n} p_{\leq k}(1 - p_{>k})h_k$ in a public order, least when investors commit in increasing order of $p_i h_i/(1 - p_i)$, so that those likely to be unwilling, or whose expense is small, commit first and a failure is discovered before others have spent much in waiting for it, and it is larger under simultaneous commitments than in any order; and when the terms consist of charges and escrowed payments no larger than expenses, willing formation in every perfect Bayesian equilibrium requires one investor to a round (Proposition 34). What an order saves is expense rather than transfer, and a risk-averse early investor is better served by an escrowed payment than by an own payment of equal expected value, since the escrow pays in the state he fears. The early refund bonuses of Cason, Tabarrok and Zubrickas (2021) are payments of the same kind; what the theorem adds is that within an order they need not exceed the expenses they insure, that a charge achieves the screening without an order, and that without either no terms paid to committing investors achieve it.

3.6 The regress and the first stage

A closed-end fund appears to escape the problem by another route, since its limited partners are bound in advance to fund capital calls, so that at the level of each investment refusal is penalized and the manager need not assure anyone, but the binding was itself a formation decision.

Corollary 14 (The regress)**.** *Let commitments be made in stages, each of which may bind investors' choices at later stages, let no investor be bound before the first, and let the first stage be a formation game of the kind studied in this section, with known willingness: a first-stage investor who declines obtains a value that does not depend on others' choices, and a first-stage committer obtains a positive gain if the first-stage commitment forms and loses his expense of standing ready if it does not. If the first-stage commitments are simultaneous, any payment scheme that implements formation at that stage robustly either includes an investor able to form the first-stage commitment alone or has liability greater than R^*. If they are made in ℓ public rounds under a given assignment*

and the scheme implements formation by backward iterated dominance, its liability is at least $\hat{R}_\ell$, and greater unless $\hat{R}_\ell = 0$.

With known willingness, a shared investment that forms robustly therefore rests, at its first stage, on an investor able to form it alone or on contingent capacity of at least R^*, which investors who believe in one another's future rationality can divide among the rounds of a calendar, and later stages can lean on penalties because the first succeeded. The capacity can be held by anyone able to pay in the state of failure, a sponsor, a bank or an insurer, and by Proposition 8 the same capacity can back many investments, so that shared contingent support can serve accounts whose ownership remains separate; nothing in the argument establishes that it must reside in a conventional fund. The companion paper, which holds the terms of formation fixed, treats simultaneous commitments under private willingness when there is a gap between indifference and unwillingness, and when there is none it shows that efficient formation by reports requires an interim commitment equal to information rents less surplus, which a sufficiently deep pool of investors eliminates and which must itself be assured at enrollment, when willingness is known again (Komkov, 2026b); Theorem 13 supplies two alternatives to reports, a charge on committing paired with an escrow and an order, and Appendix B.3 extends Theorem 3 to a closed cohort whose members know only their own gains and face a common funding risk under risk aversion, characterizing the terms under which every investor signs at every type (Theorem 36).

Binding in advance is worth its cost when the investments are many and each needs many investors, and the comparison can be made exact. Let the same n investors, all indispensable, be needed for each of K investments, let each be willing to join a given investment with probability $p \in (0, 1)$, independently across investors and investments, gaining $g > 0$ if willing and losing $\lambda > 0$ if not, and let $h > 0$ be the expense of standing ready. Deal by deal, separately owned accounts form an investment only with the consent of all, and an order with escrowed payments slightly below expenses achieves willing formation while wasting $W_n = h \sum_{k=1}^{n-1} p^k(1 - p^{n-k})$ per investment, less than simultaneous commitments waste (Theorem 13(iii) and Proposition 34). A *blind pool* binds the investors to all K investments before their willingness is known, at a formation cost $F \geq 0$ incurred once, so that every investment is made.

Proposition 15 (Blind pool)**.** *With risk-neutral investors, the blind pool yields more expected surplus than deal-by-deal formation if and only if $K\,D(n) > F$, where*

$$D(n) = n[pg(1 - p^{n-1}) - (1 - p)\lambda] + W_n.$$

A single investor never gains from pooling, since $D(1) = -(1 - p)\lambda$. If $pg > (1 - p)\lambda$, which for risk-neutral investors is also what makes each investor willing to join the pool, $D(n)$ is positive and increasing in n for every n at least $\hat{n}$, the least $n \geq 2$ with $pg(1 - p^{n-1}) \geq (1 - p)\lambda$.

Deal by deal, an investment that needs n investors forms with probability p^n and wastes W_n, and both worsen as the syndicate grows, whereas the pool gives up each investor's veto, worth $(1 - p)\lambda$ per investment, in exchange for forming every investment once and for all. With thirty

investors each willing with probability 0.99, a syndicate assembled deal by deal forms only 74 percent of the time. The proposition compares the pool with one particular alternative, unanimous consent by the same investors to every investment, and Corollary 21 supplies a third arrangement that keeps advance commitment while restoring individual exposure, in which units go where investors prefer at a priced cost of moving them. Risk aversion narrows the pool's advantage: an investor with constant absolute risk aversion α joins only if $pe^{-\alpha g} + (1-p)e^{\alpha\lambda} < 1$, which with $p = 0.9$, $g = \lambda = 1$ and $\alpha = 3$ fails although the expected gain is 0.8, and independent repetition across investments does not relax it. With $p = 0.9$, $g = \lambda = 1$ and $h = 0.2$, pooling loses for one or two investors and wins for three or more, $D(2) = -0.002$ and $D(3) = 0.263$. The proposition predicts co-investments and club deals where few investors are needed and blind pools where many are, and it is the formation counterpart of Axelson, Strömberg and Weisbach (2009), in whose model ex ante pooling disciplines a manager: here it overcomes the unanimity of separate consent, at the price of the vetoes that consent provides, and the pool's own formation is the single formation problem of Corollary 14, which requires contingent capacity or an order once rather than for every investment.

Information does not substitute for either instrument, capacity or an order. If no investor can form the investment alone and a failed subscription yields no positive net reward, universal refusal is a Bayesian equilibrium under every information structure, since a subscription that cannot form the investment cannot raise the subscriber's payoff whatever he knows (Proposition 35 in Appendix B.2). Sharper signals, disclosed willingness and recorded indications of interest therefore do not remove the equilibrium of refusal, and because no investor can form a fixed-threshold investment whatever he learns, the dominance regions from which global-games arguments select a unique equilibrium (Morris and Shin, 2003; Sákovics and Steiner, 2012) are absent; what removes the equilibrium is a change in incentives or in the order of moves, which the instruments of this section supply.

4 The book: commitment without common exposure

Section 3 priced the support on which formation depends. A pooled fund also delivers principal, gives its manager discretion to place it and gives its investors proportional exposure to whatever results, and this section separates the three and then asks which investors remain willing to supply the units a book requires once its terms are known. The object throughout is a *book*: a set of investors whose commitments bind if the book closes, a set of projects, and the permissions and allocation rules that connect them, so that the allocation of the committed capital, and with it each investor's exposure, need not be proportional.

4.1 Permissions and compatible capital

Let investors $i \in N$ commit capital $q_i > 0$, binding if the book closes, and let projects $j \in \mathcal{J}$, of which there are m, require capital $K_j > 0$ each. Investor i's *permissions* are the set $\Gamma_i \subseteq \mathcal{J}$ of

projects his capital may finance; they are written into his commitment and fixed before the book closes, so that they are contractual and verifiable rather than reported. An allocation $x = (x_{ij}) \geq 0$, with $x_{ij} = 0$ unless $j \in \Gamma_i$ and $\sum_j x_{ij} \leq q_i$, *finances* the program if $\sum_i x_{ij} = K_j$ for every project. Investor i's exposure is the row x_i, and nothing requires the rows to be proportional to one another. For a set of projects $\mathcal{J}'$, write $K(\mathcal{J}') = \sum_{j \in \mathcal{J}'} K_j$ and $\mathcal{N}(\mathcal{J}') = \{i : \Gamma_i \cap \mathcal{J}' \neq \emptyset\}$ for the investors permitted to finance at least one of them.

Proposition 16 (Compatible capital)**.** *With divisible allocations, the program can be financed if and only if*

$$K(\mathcal{J}') \;\leq \sum_{i \in \mathcal{N}(\mathcal{J}')} q_i \qquad \text{for every } \mathcal{J}' \subseteq \mathcal{J}.$$

The least additional capital, permitted to finance every project, that makes the program financeable is

$$B^* = \max_{\mathcal{J}' \subseteq \mathcal{J}} \Big[K(\mathcal{J}') - \sum_{i \in \mathcal{N}(\mathcal{J}')} q_i \Big]_+,$$

and B^* weakly falls when an investor's permissions widen or an investor is added.

The proposition is Gale's (1957) feasibility theorem for flows, the divisible form of Hall's (1935) condition, and its content for finance is that aggregate capital is the wrong measure of a population's capacity to invest. What matters is capital whose permissions jointly cover the requirements, including when projects compete for the same investors. Two projects requiring one unit each cannot both be financed by two investors of one unit each if both authorize only the first, although there is capital enough in aggregate, and $B^* = 1$; if one of them also authorizes the second, $B^* = 0$, although nothing about the projects, the capital or the forecasts of returns has changed except the compatibility of the commitments. The value of an additional investor or an additional permission is therefore what it makes feasible for everyone else: an investor whose mandate duplicates capital that is already abundant does not lower B^*, while one whose permissions cover the binding set of projects does. B^* measures missing principal and R^* missing assurance, and the two are different quantities with different remedies.

Commitment support depends on permissions in the same way, and on exposure only through willingness. Let the book close as a single event, exactly when the committed investors' capital can finance the program under their permissions, let the allocation at each closing set be chosen by a fixed rule, and let a committed investor lose his expense of standing ready h_i if the book does not close. Call an investor *willing* under the rule if his gain from every closing at which he is a member is positive, and let the book be offered to the set $\mathcal{W}$ of willing investors, whose decisions to commit then form a formation game of Section 3 with structure

$$\mathcal{F}(\Gamma, \mathcal{W}) = \{S \subseteq \mathcal{W} : \text{ the investors in } S \text{ alone can finance the program}\},$$

which is closed under enlargement within $\mathcal{W}$, so that Theorems 3 and 4 apply to it.

Proposition 17 (Commitment depends on permissions, and on exposure only through willingness)**.** *Suppose the willing investors can finance the program. The infimum of the liabilities of payment schemes that make the book's closing the unique rationalizable outcome is* $R^*(\Gamma, \mathcal{W}) = \min_{S \in \mathcal{F}(\Gamma,\mathcal{W})} \left(\sum_{i \in S} h_i - \max_{i \in S} h_i \right)$*, and escrowed schedules approach it.*

(i) The allocation rule affects $R^*(\Gamma, \mathcal{W})$ *only through the set* $\mathcal{W}$*: rules under which the same investors are willing give the same value, and a rule under which every investor willing under another rule is also willing gives a weakly smaller one.*

(ii) For a given set $\mathcal{W}$ *of willing investors, widening any investor's permissions weakly enlarges* $\mathcal{F}(\Gamma, \mathcal{W})$ *and weakly lowers* $R^*(\Gamma, \mathcal{W})$*.*

(iii) If the willing investors' committed capital equals total requirements, each of them is indispensable, and $R^*(\Gamma, \mathcal{W}) = \sum_{i \in \mathcal{W}} h_i - \max_{i \in \mathcal{W}} h_i$ *for all permissions under which the program can be financed.*

The need for collective commitment does not, by itself, establish a need for proportional exposure to a common portfolio, and the dependence that does exist runs the other way. Proportional exposure places part of every investor's capital in projects he would not choose, and an investor whose mandate excludes one of them, or whose loss from holding it exceeds his gain from the rest, is unwilling to join a pool on those terms, whereas an allocation that respects his preferences can make him willing, and if it leaves willing every investor who was willing in the pool, part (i) shows that it can only lower the cost of commitment support. Among the investors who are willing, a pool in which every investor permits every project has the largest formation structure and the least R^*, and narrower permissions weakly raise it by making investors indispensable to particular projects. Collective commitment therefore requires support whose size is fixed by permissions, willingness and expenses of standing ready. The proposition takes the willing set as given, and Section 4.4 derives it for a book whose terms are set by the mechanism that allocates its places; the question that comes first, how much breadth of permission a population must grant and at what price, is the question of discretion.

4.2 How much discretion

Let n investors each commit one unit, and let the m projects require $K_j = \rho_j n$ units, with $\sum_j \rho_j = 1$, so that all committed capital is needed. A share f of the investors are *flexible* and permit every project, and the others are *narrow* and permit one. Narrow mandates are exchangeable: conditionally on a random vector ψ of shares on the simplex, each narrow investor permits project j with probability ψ_j, independently of the others, which by de Finetti's theorem is the general form of mandates that are exchangeable in an arbitrarily large population. Mandates are *idiosyncratic* if $\psi = \rho$ with probability one, so that they differ only through independent draws centered on the requirements, and *systematic* otherwise. Let n_j be the number of narrow investors permitting j. Because all capital is needed, narrow capital in excess of a project's requirement cannot be used elsewhere, and the program is financed exactly when no project is over-subscribed by narrow

capital, $n_j \le K_j$ for every j. Let $f_n(\delta)$ be the least flexible share that finances the program with probability at least $1-\delta$, and let $z_{1-\delta/2}$ be the standard normal quantile.

Theorem 18 (Mandate diversification)**.** *(i) With idiosyncratic mandates and two equal projects, $\sqrt{n}\, f_n(\delta) \to z_{1-\delta/2}$. With m projects, flexible capital $fn \ge \sqrt{8n\log(m/\delta)/(3\rho_{\min})}$ finances the program with probability at least $1-\delta$, where $\rho_{\min} = \min_j \rho_j$.*

(ii) If $Y = \min_j \rho_j/\psi_j$, which never exceeds one, has a distribution function that is continuous and strictly increasing at its δ-quantile Q_δ, then $f_n(\delta) \to f^*(\delta) = 1 - Q_\delta$. With two equal projects $f^*(\delta) = \zeta_\delta/(1+\zeta_\delta)$, where ζ_δ is the $(1-\delta)$-quantile of $|2\psi_1 - 1|$.

(iii) If each narrow mandate follows, with probability η, a common recommendation of a given project r, and is otherwise idiosyncratic, then $f_n(\delta) \to \eta(1-\rho_r)/(\eta + (1-\eta)\rho_r)$ for every δ, which is $\eta/(1+\eta)$ with two equal projects.

(iv) If narrow investors come in blocks of k that share one mandate, the requirement with two equal projects is that of n/k investors of k units each, and $\sqrt{n}\, f_n(\delta) \to z_{1-\delta/2}\sqrt{k}$.

Differences among mandates diversify as returns do, up to a systematic component that does not. When each investor's restriction is an independent draw centered on what the projects require, the imbalance between narrow commitments and requirements grows only with the square root of the population, so that flexible capital of order $\sqrt{n}$ absorbs it and its share of the whole vanishes: with ten thousand investors, 254 flexible units, about two and a half percent of the capital, finance both of two equal projects with probability 99 percent, while the remaining 97.5 percent keeps its narrow restriction. When mandates share a component, through a common view, a common adviser or a common recommendation, the imbalance grows in proportion to the population and so does the discretion it requires; if one mandate in five follows a single recommendation between two equal projects, a sixth of all capital must be flexible however many investors there are. Dividing one investor's commitment among many accounts creates no independence, since what diversifies is the number of independent decisions, not the number of accounts. The square-root law is that of inventory pooling (Eppen, 1979) and of staffing at critical load (Halfin and Whitt, 1981), the persistence of a common component is that of pooling under dependent demand (Corbett and Rajaram, 2006), and the sufficiency of a little flexibility is known from process flexibility and resource pooling (Jordan and Graves, 1995; Bassamboo, Randhawa and Van Mieghem, 2012; Tsitsiklis and Xu, 2012); the content for finance is that the permissions of a population of investors are a dimension of diversification distinct from returns, and that their systematic part, not their idiosyncratic part, determines how much discretion a population must surrender. Two assets whose returns are nearly independent can be excluded by the same investors for the same reasons, so that diversification in returns does not imply diversification in permissions.

The benchmark places the book at critical load, with committed capital equal to requirements, which is the case of a purchase of fixed size allocated in full among the investors committed to it. Committed capital exceeding requirements by a margin of order $\sqrt{n}$ would substitute for flexibility, as spare capacity does in a queue, but unused commitments bear expenses of standing

ready, and with such a margin permissions also matter for commitment support, since they then determine which sets of investors can finance the program, whereas at critical load every willing investor is indispensable and the two questions separate. The benchmark also fixes permissions and requirements before the imbalance is known; an arrangement that adjusted mandates to the opportunities realized, or resized the projects to demand, as book-building does for offerings, would need less discretion than the theorem requires.

4.3 The price of discretion

Breadth of permission is costly to the investor who grants it, and Theorem 18 counts flexible capital as though it were free. Let each investor favor a project $j(i)$ and bear a cost $c_i \geq 0$, in certainty-equivalent dollars, if his unit is placed in any other, and c_i times the fraction placed elsewhere if the unit is divided, with $c_i = \infty$ for a narrow investor, so that a flexible investor is one whose cost is finite. A linear cost of a divided unit favors no arrangement in particular; investors who value diversification across projects would lose less from a divided unit, and Section 5 treats that case through the cost of exposure profiles. Favored projects and costs are private, and requirements are integers, as they are in Theorem 18 when each $\rho_j n$ is. A mechanism assigns each unit to a project so that every requirement is met, and it may make transfers. Let n_j be the number of investors favoring j, $D_j = (n_j - K_j)_+$, and $c_{(l)}^{(j)}$ the l-th smallest cost among them.

Theorem 19 (The price of discretion)**.** *(i) If each over-subscribed project has at least D_j supporters with finite costs, the least total cost of an assignment that meets every requirement is $C_n = \sum_j \sum_{l \leq D_j} c_{(l)}^{(j)}$, attained by moving the D_j supporters with the smallest costs off each over-subscribed project. C_n is also the least cost over fractional assignments, and it never exceeds the cost $\sum_i (1 - \rho_{j(i)}) c_i$ of pro rata exposure.*

(ii) The Vickrey–Clarke–Groves mechanism implements such an assignment in dominant strategies. With two projects, let A be the over-subscribed project, if either is: the K_A places in A go to the investors reporting the highest value of A over B, at a price ϖ_n equal to the highest unsuccessful report, which is the cost of the marginal investor moved to B; places in B are free, and nobody pays when neither project is over-subscribed.

(iii) Let favored projects be drawn independently with probabilities ρ, let each investor be flexible with probability $\varphi \in (0, 1]$, independently, and let flexible investors' costs be drawn independently from a distribution G_c with finite mean. The probability that the program can be financed tends to one, and the expected least cost on that event is at most $\mathbb{E}[c] \sum_j \sqrt{n\rho_j(1 - \rho_j)}$. If G_c has a density g_c that is positive and continuous at zero, the expected least cost on that event converges to $(m - 1)/(4\varphi g_c(0))$, and with two projects $\sqrt{n}\, \varpi_n$ is bounded in probability. If every investor is flexible, the expected cost of pro rata exposure is $n\, \mathbb{E}[c] \sum_j \rho_j(1 - \rho_j)$.

(iv) If every investor is flexible, costs are as in (iii) with $G_c(0) = 0$, and favored projects are conditionally independent given a share vector ψ that differs from ρ with positive probability, then $\mathbb{E}[C_n]/n$ converges to $\mathbb{E}[\sum_j \psi_j \int_0^{u_j} G_c^{-1}(u)\,du] > 0$, with $u_j = (1 - \rho_j/\psi_j)_+$.

(v) With two equal projects and the hypotheses of (iii), including $g_c(0) > 0$, the number of investors moved, the premium and the least cost satisfy

$$\Big(\frac{D_n}{\sqrt{n}},\ \sqrt{n}\,\varpi_n,\ C_n\Big) \Rightarrow \Big(\frac{|Z|}{2},\ \frac{|Z|}{\varphi g_c(0)},\ \frac{Z^2}{4\varphi g_c(0)}\Big)$$

jointly in distribution, where D_n is the over-subscription and Z is a standard normal variable, the event that the program cannot be financed having probability tending to zero; moreover, conditionally on the event of financing, $\sqrt{n}\,\mathbb{E}[\varpi_n] \to \mathbb{E}|Z|/(\varphi g_c(0))$. With costs uniform on $[0, 1]$ and every investor flexible, $\mathbb{E}[C_n] = n/(4(n + 1))$ for every even n.

The expected total cost of the discretion a book uses therefore stays bounded as the population grows when mandates are idiosyncratic and a positive share of investors accepts reassignment at a price. An efficient book moves only the investors needed to balance the projects, of order $\sqrt{n}$ in number, and it moves the cheapest of them; when the costs of a population of flexible investors have a positive density at zero, so that the share with costs below c is proportional to c for small c, the cheapest $\sqrt{n}$ of them cost of order $1/\sqrt{n}$ each, and the product is of order one, whereas with a density that vanishes at zero the total grows without bound, although more slowly than $\sqrt{n}$. With costs uniform between zero and a scale $\bar{c}$ and two equal projects, the expected total converges, as the book grows, to $\bar{c}/(4\varphi)$, and with every investor flexible it equals $\bar{c}\,n/(4(n + 1))$ exactly for even n, while pro rata exposure, feasible only when every investor is flexible, which places half of every investor's capital in the project he likes less, costs the average investor $\bar{c}/4$, so that its total grows in proportion to the population. The bound needs both conditions, for if moving any investor costs at least $c_0 > 0$, as a fixed cost of documentation would, the total grows like c_0 times the number moved, of order $\sqrt{n}$, although its share per investor still vanishes; and if only $a\sqrt{n}$ investors are flexible, which by Theorem 18 suffices, for a large enough, to finance the program with any given reliability, the investors moved are a fraction of the flexible ones that does not vanish, their average cost does not fall, and the total again grows like $\sqrt{n}$. The flexible capital needed to finance a book is thus of order $\sqrt{n}$, while the flexible population needed to finance it cheaply is of order n, so that a pool of discretion should be broad and shallow, with many investors granting priced permission and few of them moved. When mandates are systematic and no cost is zero the cost per investor is positive and does not vanish, and it is the price of the systematic component that Theorem 18 isolates. The premium ϖ_n is a transfer rather than a cost: it is of order $1/\sqrt{n}$, and the total collected for places in the crowded project, of order $\sqrt{n}$, is paid by investors who obtain the places they favor.

Scale bounds the expected cost of discretion without making the realized cost predictable. By part (v), one Gaussian imbalance determines the number moved, the premium and the cost together, and the realized cost converges not to a constant but to a multiple of a chi-squared variable with

one degree of freedom, whose 99th percentile is 6.6 times its mean, so that a book of any size will occasionally incur a misallocation cost several times its average. The distribution describes economic losses, not a liability: it becomes a claim on someone only if a contract compensates investors for being moved, and it is distinct from the capacity of Section 3, which pays when formation fails, and from the replacement capital that Section 4.4 shows a book may need.

The form of the mechanism matters as well as its cost, since in the uniform-price assignment only the crowded project carries a price, an investor's report affects his payment only through the project he is assigned, and Vickrey's (1961) argument makes truthful reporting dominant; with more than two projects the Vickrey–Clarke–Groves payments are the least Walrasian prices of the assignment game (Shapley and Shubik, 1971; Leonard, 1983; Demange, Gale and Sotomayor, 1986), and the arrangement is a uniform-price book-building for places, in the spirit of the allocation of new issues (Benveniste and Spindt, 1989). The premium accrues to whoever operates the assignment. Returning it in equal shares to all the book's investors balances the budget and gives up exact incentive compatibility for a gain from misreporting of the order of the share K_A/n of the gap between the reports adjacent to the margin; returning it instead to the investors moved also balances the budget but does not keep the gain small, since each would receive about $1/(\varphi g_c(0))$, far more than his cost, which is at most ϖ_n, and this would recreate the incentive described next. That no efficient mechanism is in general both incentive compatible in dominant strategies and budget balanced is the theorem of Green and Laffont (1977). The alternative to pricing the crowded project is to pay flexible investors for being moved.

Proposition 20 (Price the crowded project)**.** *Let there be two projects, let favored projects be declared when commitments are made, let costs have a distribution without atoms, and let the other investors' declarations alone over-subscribe project A. A mechanism that pays investors to move off their declared project, at the cost of the marginal investor moved, gives an investor who is indifferent between the projects a positive expected gain from declaring A. Under the uniform-price assignment no investor gains by misstating his favored project or his cost.*

An arranger who pays for flexibility invites investors to declare the crowded project in order to be paid for leaving it, which inflates the imbalance the payment was meant to absorb, whereas one who charges for the crowded project leaves no such incentive, so that the principle is to price the scarce position rather than to buy the flexible one. The same comparison governs the supply of flexibility. A narrow mandate is a report of an infinite cost, so that under the uniform-price assignment it is weakly dominated for every investor whose cost is finite, flexibility adding only the option of being moved when the premium exceeds his cost. Under a rule that moves flexible investors without compensation, a narrow mandate is the better choice whenever others can absorb the shortage it creates and the worse one when it would prevent the book from closing, as when an investor who gains 2 from formation and would bear a cost of 1 in his less favored project is the only one able to finance it; the supply of flexibility then becomes a formation problem of the kind studied in Section 3, in which each investor prefers that others supply it.

Corollary 21 (Book and pool)**.** *Under the hypotheses of Theorem 19(iii), with g_c positive and continuous at zero, a book in which the uniform-price assignment or its generalization allocates units finances every project with probability tending to one, at an expected misallocation cost on the event of financing that converges to $(m-1)/(4\varphi g_c(0))$. A blind pool, which can admit only investors willing to hold every project, finances every project only if every investor is flexible, and then at an expected cost of $n\,\mathbb{E}[c]\sum_j \rho_j(1-\rho_j)$; deal-by-deal formation, in which each investor finances only his favored project, finances every project with probability tending to zero. Under systematic mandates, with every investor flexible, both costs grow in proportion to n, and the book's never exceeds the pool's. When every investor permits every project, the book and the pool have the same formation structure among the investors willing in both.*

The book thus combines the reliability of the blind pool with, asymptotically, the selectivity of separate consent. A blind pool finances everything because every investor's capital can go anywhere, and it pays for that breadth with exposure that ignores what each investor wants and with the exclusion of investors whose mandates forbid some project. Separate consent respects every preference and finances a program only if preferences happen to match requirements exactly. A book in which many investors grant priced permission finances everything the pool finances, admits narrow investors whom the pool cannot, and, when differences in mandates are idiosyncratic, uses discretion whose expected total cost does not grow with the population. The comparison holds fixed the population of potential investors and their capital rather than the capital committed voluntarily, which differs between the arrangements because the pool admits only investors willing to hold every project and the book admits others; Section 4.4 asks which of them commit. A strict pro rata pool is, moreover, the benchmark rather than the strongest alternative. Funds that grant excuse rights releasing an investor from particular investments, side letters that exclude sectors, and co-investment vehicles alongside the fund already let some investors decline particular investments and others take more of them, so that they are partial books, and the comparison that matters in practice is between such arrangements and a book with priced places, at equal costs of administration and records (Section 5).

The comparison also says where each arrangement belongs. Where mandates differ idiosyncratically, the book's expected cost of discretion stays bounded while the pool's grows in proportion to the population, so that the book's advantage per investor does not vanish and the theory predicts the book. Where a common component moves a share of mandates together, the flexible share the book needs converges to a positive limit (Theorem 18(iii)) and its expected cost per investor to a positive limit (Theorem 19(iv)), so that both costs grow in proportion to n, the book's smaller than the pool's by a factor that depends on the dispersion of costs; the cost of the records and incentive pay that common exposure economizes is proportional to the population as well (Section 5), and whether the book's smaller cost of discretion pays for the records it keeps separate is then a comparison of two quantities of the same order that the paper does not make. A manager whose judgment moves all of his investors' mandates together is the limiting case of a common component, and the fund is the arrangement the theory expects around him.

4.4 Forming the book: the right to decide later

Theorem 19 allocates units that investors have already committed, and Proposition 17 takes as given the investors who are willing to commit them, so that neither determines which investors still supply the units a book requires once places are priced and some investors are moved, which is the question on which an institution built on them depends. Let two projects require $K = n/2$ units each, with n even, and let every investor be flexible. Investor i obtains $v_i > 0$ from a place in the project he favors, relative to staying out, $v_i - c_i$ from a place in the other, and nothing if he is not placed or the book does not close, less any price he pays and plus any share of the premium he receives. Margins are net of the expenses of standing ready, which are set aside here because Section 3 prices them, so that a book that does not close costs no one anything. Each investor favors each project with probability one half, and favored projects, costs and margins are independent across investors; costs are independent of favored projects and drawn from a distribution G_c with finite mean and a density g_c that is positive and continuous at zero; margins are independent of favored projects and costs and drawn from a distribution G_v on $(0, \infty)$ with a density g_v that is positive and continuous at zero; and investors are risk neutral. Places are allocated by the uniform-price assignment of Theorem 19(ii), whose premium is either retained by the arranger or returned in equal shares to the holders of the n places, and the book holds *replacement capital* of S_n units. Two arrangements differ only in the moment at which an investor's participation becomes binding.

Under a *retained veto*, the n investors report their favored projects and costs, the assignment and its price are announced, and each then accepts his place or withdraws; replacement capital, committed in advance to take any vacated place on that place's terms, fills up to S_n vacancies, and the book closes if no more than S_n investors withdraw. Under *binding enrollment*, $n + S_n$ investors are invited before favored projects and costs are known, each knowing only his own margin; an investor who enrolls is bound to take any place the assignment gives him on its terms, the book closes if at least n enroll, in which case n of them, chosen at random, are placed and the others are released at no cost, and reports are made after enrollment, so that replacement capital here consists of the invitations beyond the n the book requires. Write Q_n for the number who withdraw under a retained veto and Q'_n for the number who decline to enroll under binding enrollment, so that the book closes exactly when the relevant shortfall is at most S_n. A placed investor's *net burden* is the price or cost he bears less any share of the premium he receives. Reports are taken to be truthful in parts (i)–(iii) and (v) of the theorem that follows, and part (iv) examines when they are. Let $\gamma = g_v(0)/g_c(0)$ be the ratio of the densities of margins and of costs at zero, and let Z be a standard normal variable.

Theorem 22 (The right to decide later)**.** *(i) Under a retained veto, withdrawing exactly when one's margin is below one's net burden weakly dominates every other rule. When placed investors follow it, $Q_n/\sqrt{n}$ converges in distribution to $\gamma|Z|/2$ if the premium is retained and to $\gamma|Z|/4$ if it is returned. If $S_n/\sqrt{n} \to a \geq 0$, the probability that the book closes converges*

to $\mathbb{P}(|Z| \le 2a/\gamma)$ *or* $\mathbb{P}(|Z| \le 4a/\gamma)$ *respectively, which is less than one for every finite* a *and zero when* $a = 0$*, as at critical load; it converges to one if and only if* $S_n/\sqrt{n} \to \infty$*.*

(ii) Under binding enrollment with the premium retained, enrolling exactly when one's margin is at least the expected net burden v_n^* *of a placed investor weakly dominates every other rule, and* $\sqrt{n}\, v_n^* \to 1/(\sqrt{2\pi}\, g_c(0))$*. When invited investors follow it and* $S_n/\sqrt{n} \to a \in [0, \infty)$*,* $Q_n'/\sqrt{n}$ *converges in probability to* $a^* = \gamma/\sqrt{2\pi}$*, and the probability that the book closes converges to one if* $a > a^*$ *and to zero if* $a < a^*$*. The limit in part (i) with the premium retained has the same mean,* $\gamma\, \mathbb{E}|Z|/2 = a^*$*.*

(iii) Under binding enrollment with the premium returned, the same rule weakly dominates every other, with $v_n^* = \mathbb{E}[C_n]/n$*. When invited investors follow it and* $S_n/n \to 0$*,* Q_n' *converges in distribution to a Poisson variable with mean* $\mu_v = \gamma/4$*, which is* $g_v(0)$ *times the limit of the expected total real cost of discretion; the probability that the book closes converges to* $\mathbb{P}(\text{Poisson}(\mu_v) \le s)$ *when* $S_n = s$ *is a fixed integer, to* $e^{-\mu_v}$ *at critical load, and to one if* $S_n \to \infty$*.*

(iv) Under binding enrollment, truthful reporting is a dominant strategy when the premium is retained, and when it is returned and $n \ge 4$ *a misreport raises an investor's payoff by at most* $\frac{1}{2}\max\{w^{-i}_{(K-1)} - w^{-i}_{(K)},\ w^{-i}_{(K)} - w^{-i}_{(K+1)}\}$*, where* $w^{-i}_{(l)}$ *is the l-th largest of the other investors' reported values of a place in A over one in B. Under a retained veto with the premium retained, the investor whose report sets the price can raise the probability that the book closes, without changing his place, by lowering his report.*

(v) Let each invited investor know his favored project and cost, as well as his margin, when he enrolls, let the premium be returned, and let the book be at critical load, $S_n = 0$*. There is a constant* C_0*, depending only on* G_c*, such that whenever the other investors enroll by a common rule that treats the two projects alike and enrolls at least half of the investors of every cost, a placed investor's expected net burden, given his own favored project and cost, is at most* C_0/n*. For all large* n *a symmetric equilibrium in such rules exists; in every such equilibrium each investor whose margin exceeds* C_0/n *enrolls, and the probability that the book closes is at least* $(1 - G_v(C_0/n))^n$*, which converges to* $e^{-g_v(0)C_0} > 0$*.*

Corollary 23 (The price of deciding later)**.** *With truthful reports and the rules of Theorem 22, let* $S_n(\delta)$*, for* $\delta \in (0, 1)$*, be the least replacement capital with which the book closes with probability at least* $1 - \delta$*. With the premium retained,* $S_n(\delta)/\sqrt{n}$ *converges to* $z_{1-\delta/2}\, \gamma/2$ *under a retained veto and to* $a^* = \gamma/\sqrt{2\pi}$ *under binding enrollment, so that their ratio converges to* $z_{1-\delta/2}\sqrt{\pi/2}$*, which is 2.5 at* $\delta = 0.05$ *and 3.2 at* $\delta = 0.01$*. With the premium returned,* $S_n(\delta)/\sqrt{n}$ *converges to* $z_{1-\delta/2}\, \gamma/4$ *under a retained veto, while under binding enrollment* $S_n(\delta)$ *converges, for all but countably many* δ*, to the least integer* s *with* $\mathbb{P}(\text{Poisson}(\mu_v) \le s) \ge 1 - \delta$*, which does not grow with the population.*

The two arrangements produce, to first order, the same expected shortfall when the premium is retained, and require very different amounts of replacement capital. Under a retained veto, the

investors who refuse are those who pay the premium for a place in the crowded project and whose margins are smaller than the price, and their number is proportional to how crowded the project turned out to be, so that every refusal is driven by the one imbalance that all investors share, a correlation that the mechanism itself creates, because preferences are drawn separately for each investor and one clearing price, applied to all of them, links decisions to participate that differences in fit alone would leave unrelated; the shortfall therefore fluctuates on the scale of its mean, and replacement capital must cover the tail of that common shock rather than its average. Under binding enrollment, each investor's decision depends only on his own margin and on an expectation that is the same for everyone, so that refusals are independent, their number concentrates, and replacement capital need only cover the mean. The right to decide after the terms are known thus has a financing cost, the replacement capital that must stand behind a decision whose outcome depends on information common to all, and its size depends on whether participation binds before or after that information arrives. The limits of the corollary are approached slowly, because under binding enrollment the shortfall fluctuates around a mean of order $\sqrt{n}$ by amounts of order $n^{1/4}$, so that the replacement capital needed for a given reliability exceeds its limit by a share that falls only like $n^{-1/4}$: with costs and margins uniform on the unit interval and $\delta = 0.01$, the exact replacement capital under binding enrollment is 43 percent of the limiting requirement under a retained veto with ten thousand investors, 38 percent with a hundred thousand and 35 percent with a million, against a limit of 31 percent. The comparison counts units of different kinds. Under a retained veto a unit of replacement capital is committed in advance to fund any vacated place, while under binding enrollment it is an invitation that binds only an investor who accepts and is released if not needed, so that a comparison of carrying costs must also price how long each unit is reserved, what its holder forgoes and whether its cash must already be available, the first of which capital multiplied by the time for which it is reserved measures, and the others the price of each unit of that capital-time.

Returning the premium within the book halves the shortfall under a retained veto without changing its order, but under binding enrollment it collapses the shortfall to a bounded number, because each investor's expected net burden is then his share of the real misallocation cost, of order $1/n$, rather than his expected premium, of order $1/\sqrt{n}$, which is a transfer within the book. A book at critical load, in which every investor is indispensable, then closes with probability tending to $e^{-\mu_v}$, where μ_v is the density of margins at zero times the limit of the expected total real cost of the book's discretion, and replacement capital that grows without bound, however slowly, makes its formation certain in the limit, which is the formation counterpart of the bounded cost in Theorem 19(iii). Part (v) shows that the conclusion does not rest on investors' ignorance of their own types. When each investor knows his favored project and cost at enrollment, expected burdens differ across types, and an investor with a high cost bears more than the average when his project turns out to be crowded, but the share of the premium he receives when the other project is crowded nearly offsets it, so that no type's expected net burden exceeds a multiple of $1/n$; with costs uniform on the unit interval and every other investor enrolled, the largest is about $1/(2n)$, twice the average, and falls on the investors with the highest costs, while investors whose costs

are low expect a net receipt. At critical load an investor's enrollment matters only if all the others enroll, so that refusal by everyone is also an equilibrium, and part (v) describes the equilibria in which most investors enroll rather than a choice that dominates.

The theorem bears on the analysis of participation constraints in mechanism design. When types are independent and participation is decided before private information arrives, an efficient mechanism can balance its budget and leave every participant willing (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979); when it is decided afterwards, efficient trade can be impossible (Myerson and Satterthwaite, 1983), and the initial distribution of ownership determines whether a partnership can be dissolved efficiently (Cramton, Gibbons and Klemperer, 1987). In large economies the inefficiency that interim participation constraints impose vanishes in some environments (Gul and Postlewaite, 1992) and not in others: in the public-good environment of Mailath and Postlewaite (1990), where the cost of provision grows in proportion to the number of members, the probability of provision under voluntary participation vanishes, whereas Hellwig (2003) shows that when the benefit and cost of a nonexcludable public good do not depend on the size of the population, second-best provision converges to the first best. Battaglini and Palfrey (2024) show that when members cannot be compelled to follow a mechanism's recommendation, so that obedience must be in each member's interest once the recommendation is made, the best mechanism for a large group achieves in the limit no higher probability of success than an unorganized group. A retained veto is an obedience constraint of this kind, imposed after the information common to all is revealed, and binding enrollment removes it. The real cost of the book is bounded, and parts (iii) and (v) show that a bounded cost, spread over all investors before the imbalance is known, leaves a probability of formation bounded away from zero, in part (v) in the equilibria described there, as Hellwig's analysis would suggest for a fixed cost. What defeats formation under a retained veto, in the uniform-price assignment, is the timing of transfers: each investor may refuse after a premium that is a transfer rather than a cost has fallen on him, and transfers of total order $\sqrt{n}$ that fall, after the common imbalance is known, on one side of it produce refusals of the same order.

Timing also governs incentives. Under binding enrollment the closing is settled before reports are made, so that the dominance argument of Theorem 19(ii) applies exactly when the premium is retained and nearly when it is returned in equal shares; under a retained veto with the premium retained, the investor whose report sets the price can raise the probability of closing by shading it, and a cancellation rule attached to an assignment mechanism changes the mechanism whose incentives it was meant to inherit. A unilateral deviation by the investor who sets the price can lower it only to the next report, a gap of order $1/n$ in probability, and changes the expected shortfall by an amount of order one, but equilibrium reporting under a retained veto, in which no investor knows in advance whether his report will set the price, is not characterized here. Part (i) concerns the uniform-price assignment, and Propositions 25 and 26 below show that, under a retained veto and when margins have a finite mean, every mechanism that makes the efficient assignment with dominant-strategy incentives draws, in expectation, on capacity of order $\sqrt{n}$, and, when they have a finite second moment, on capacity of that order for any reliability above one half. An assignment

that moved the cheapest investors without charging for the crowded project would put a burden of order $1/\sqrt{n}$ on each of order $\sqrt{n}$ investors and lose a bounded number of them, but it would give no investor a reason to report a finite cost, since a narrow mandate would spare him the move (Proposition 20); in the uniform-price assignment, the price that makes reports truthful is also the transfer that makes refusals numerous.

The bound on the gain from misreporting in Theorem 22(iv) is of order $1/n$ in probability, which is also the order of the expected net burden that produces the bounded shortfall of its part (iii), so that a small incentive to misreport could in principle change the limiting constant. A rebate computed as though the investor had not reported removes the incentive altogether. It is the redistribution of Bailey (1997) and Cavallo (2006), applied to the uniform-price assignment.

Proposition 24 (Rebates that leave out the investor's own report)**.** *Under binding enrollment, let the premium be returned by paying the holder of each place K/n times the price ϖ_n^{-i} that the uniform-price assignment would set if his own report were removed, and let $\Lambda_n = K\varpi_n - \sum_i (K/n)\varpi_n^{-i}$ be the part of the premium that is not returned.*

(i) Truthful reporting is a dominant strategy, and $0 \le \Lambda_n \le K\varpi_n$, so that the rebates never exceed the premium.

(ii) With truthful reports and under the hypotheses of Theorem 22, Λ_n converges in distribution to an exponential variable with mean $1/(2g_c(0))$, jointly with the limits of Theorem 19(v) and independently of Z, and $\mathbb{E}[\Lambda_n] \to 1/(2g_c(0))$.

(iii) With truthful reports, enrolling exactly when one's margin is at least $\mathbb{E}[C_n + \Lambda_n]/n$ is optimal whatever the others' enrollment decisions, so that every investor's following this rule and reporting truthfully is an equilibrium of the whole game of enrollment and reporting. If $S_n/n \to 0$, Q_n' converges in distribution to a Poisson variable with mean $3\mu_v = 3\gamma/4$, and the probability that the book closes converges to $e^{-3\gamma/4}$ at critical load and to $\mathbb{P}(\text{Poisson}(3\gamma/4) \le s)$ when $S_n = s$ is a fixed integer.

(iv) Part (v) of Theorem 22 holds under this rebate with a larger constant in place of C_0, truthful reports being dominant by part (i) rather than assumed.

Exact incentives cost a premium that cannot be returned. Only the $K + 1$ investors whose removal would lower the price receive less than they would under equal shares, and each receives less by K/n times the gap between the price and the next lower cost among the supporters of the crowded project, a gap of order $1/n$, and the price itself when only one investor is moved. The shortfall is therefore $K(K + 1)/n$ times that gap, a bounded transfer whose mean, $1/(2g_c(0))$, is twice the limit of the expected real cost of the book's discretion. It triples the Poisson mean and lowers the limiting probability of closing at critical load from $e^{-\gamma/4}$ to $e^{-3\gamma/4}$, from 78 to 47 percent when $\gamma = 1$, while two or three additional invitations raise it above 95 or 99 percent. In multi-unit auctions with unit demand, which the places in a crowded project resemble, rebates built from more of the other bidders' reports return more of the payments without giving up dominance (Moulin, 2009; Guo and Conitzer, 2009), so that the factor of three belongs to the simplest exact

rule rather than to exact incentives as such; how much of the premium such rules can return here is not settled, but the simplest already keeps the shortfall bounded. Because no investor in the book receives it, the premium that is not returned can pay for standby capacity supplied by parties outside the book without disturbing the incentives inside it. The total net burden of the placed investors, the cost of the moves plus the premium that is not returned, converges in distribution to $\chi^2_3/(4g_c(0))$, whose 99th percentile is 3.8 times its mean against 6.6 times for the cost of the moves alone, so that the rebate triples the expected burden while thinning its upper tail relative to its mean (Corollary 41 in Appendix B.4); the sum measures what the book takes from its investors, part of it as a transfer, and is not a liability to be reserved against.

Proposition 24 concerns binding enrollment, while Theorem 22(i) shows that under a retained veto the uniform-price assignment loses a number of investors of order $\sqrt{n}$ whether its premium is retained or returned, and whether other transfers could avoid that loss depends on the incentives they are required to provide. With the projects labelled A and B in advance, let ϖ_i° be the *price that investor i faces* at a profile of reports, the amount by which the others' least total cost rises when his unit is placed in A rather than in B, which is negative when his placement in A spares the others a move, so that the assignment of Theorem 19(i) places him in A exactly when his value of a place in A over one in B exceeds ϖ_i°; say that A is *crowded without i* when $\varpi_i^\circ > 0$ and B when $\varpi_i^\circ < 0$. Under a retained veto, let a mechanism take reports of favored projects and costs, make that assignment, and announce with it a transfer for each placed investor, paid by him when positive and to him when negative, that depends only on the reports, has a finite expectation and is paid only if he accepts; and let replacement capital fill every vacated place without transfers, rather than on that place's terms as in Theorem 22, so that the book closes and Q_n counts the places that replacement capital fills. Write T_n for the sum of the transfers paid by the investors who accept, those paid to them entering negatively, and $\Delta_n = (-T_n)^+$ for the cash the mechanism must find in a state beyond what it collects, so that a closing draws on capacity of two kinds, the Q_n places that replacement capital fills and the Δ_n of cash that stands behind the mechanism's own payments. The mechanism's transfers make truthful reporting a *dominant strategy in the assignment* if, for every favored project and every cost, and whatever the others report, a truthful report minimizes an investor's net burden, his cost of the place assigned plus his transfer, and since replacement capital fills every vacancy, truthful reporting is then dominant under the veto as well. The mechanism is *unsubsidized in expectation* if $\mathbb{E}[T_n] \geq 0$, so that neither the replacement capital nor anyone else outside the book finances it on average. A charge x imposed on every investor drives away the share $G_v(x)$ of them and collects $x(1 - G_v(x))$ per investor, and

$$\kappa_v = \inf_{x>0} \frac{G_v(x)}{x\,(1 - G_v(x))},$$

with the ratio taken to be infinite where $G_v(x) = 1$, is the least rate at which a charge loses investors per unit of revenue it raises. The ratio tends to $g_v(0)$ as x falls to zero, so that $\kappa_v \leq g_v(0)$; $\kappa_v > 0$ exactly when $x(1 - G_v(x))$ is bounded, and in particular whenever margins have a finite mean, and $\kappa_v = g_v(0)$ when margins are uniform or exponential. Let $\lambda_v = \min\{\kappa_v, g_v(0)/2\}$.

Proposition 25 (Incentives under a retained veto)**.** *Under the hypotheses of Theorem 22, let placed investors withdraw exactly when the margin is below the net burden, as in part (i) of that theorem, and let a mechanism of the kind just described allocate the places.*

(i) If its transfers make truthful reporting a dominant strategy in the assignment, then, with truthful reports,

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Q_n] + \lambda_v\, \mathbb{E}[\Delta_n]}{\sqrt{n}} \;\geq\; \min\Big\{\frac{2\kappa_v}{g_v(0)},\, 1\Big\}\, \frac{\gamma\, \mathbb{E}|Z|}{4},$$

and if the mechanism is unsubsidized in expectation the same bound holds for $\mathbb{E}[Q_n]$ alone. For some $\tau_0 \geq 0$, the uniform-price assignment with the rebate of Proposition 24, less a fee of τ_0/n for each place, makes truthful reporting dominant in the assignment and is unsubsidized in expectation for all large n, and under it $Q_n/\sqrt{n}$ converges in distribution to $\gamma|Z|/4$, $\mathbb{E}[Q_n]/\sqrt{n}$ to $\gamma\,\mathbb{E}|Z|/4$, and Δ_n is bounded in probability.

(ii) Let $\bar{\varpi}_n^\circ(c) = \mathbb{E}\big[\varpi_1^\circ \mathbf{1}\{\varpi_1^\circ > c\}\big]$ for $c \geq 0$, the expectation of the price an investor faces when the others report truthfully, counted only when it exceeds c, and let each placed investor i who accepts pay $\frac{1}{n-1}\sum_{k\neq i} \bar{\varpi}_n^\circ(\hat{c}_k) - \bar{\varpi}_n^\circ(\hat{c}_i)$, where $\hat{c}_k$ is investor k's reported cost. These transfers sum to zero over the placed investors, and truthful reporting is a Bayesian equilibrium when participation binds. Under a retained veto with truthful reports, Q_n converges in distribution, jointly with the limits of Theorem 19(v), to a Poisson variable whose mean is itself random and equal to

$$\frac{\gamma}{2}\Big(1 + \int_0^{|Z|} \Big(s - \frac{e^{-s^2/2}}{\sqrt{2\pi}}\Big)^+ ds\Big),$$

whose expectation is 0.659γ. With a fee of τ_0/n for each place added to these transfers, for a fixed $\tau_0 > 0$, truthful reporting remains a Bayesian equilibrium when participation binds, the mechanism is unsubsidized in expectation under the veto for all large n, and the limit holds with $g_v(0)\tau_0$ added to its mean.

(iii) Under a retained veto and for all large n, truthful reporting is not a Bayesian equilibrium of the mechanism of part (ii), with or without the fee. For each $c > 0$ there are $\underline{v}_n$ with $\sqrt{n}\,\underline{v}_n \to 1/(\sqrt{2\pi}\, g_c(0))$ such that, for all large n, when the others report truthfully, an investor whose cost is at least c and whose margin is below $\underline{v}_n$ gains by reporting a cost of zero, and the expected number of such investors, divided by $\sqrt{n}$, converges to $(1 - G_c(c))\,\gamma/\sqrt{2\pi}$.

The bound of part (i) is informative when $\kappa_v > 0$, and some such condition is needed, since when $x(1 - G_v(x))$ is unbounded a mechanism can raise expected revenue of order $\sqrt{n}$ by a fee on a single investor, at the risk of losing only him, and pay every investor placed in the project that is not crowded without him the price he faces, charging the others nothing, so that no one else refuses. Whatever the others report, an investor's transfer can depend on his own report

only through the project in which he is placed, and the two transfers must differ by the price he faces, which is the form that Green and Laffont (1977) and Holmström (1979) establish for efficient mechanisms with dominant-strategy incentives and an instance of the payment identity of Myerson (1981), so that, among investors who are not moved, one who favors the project that is crowded without him bears that price more than one who favors the other. Transfers that depend on the others' reports can move both net burdens together but cannot narrow the gap between them, and a mechanism that lowered both far enough that neither side refused would have to pay amounts of the order of the price to about half of its investors, which it can finance only by charges that drive others away or by cash from outside the book; the inequality of part (i) trades the two at the rate λ_v, so that a dollar the mechanism must find counts as λ_v places, which is half the density of margins at zero when a charge loses investors at no less than half the rate at which small charges do, $\kappa_v \geq g_v(0)/2$. In that case the bound equals the limit of the expected shortfall under the leave-one-out rebate, so that the loss of order $\sqrt{n}$ in Theorem 22(i) is not peculiar to the uniform-price assignment: among mechanisms unsubsidized in expectation that make the efficient assignment with dominant-strategy incentives, that rebate, with a fee where one is needed, is asymptotically the best a retained veto permits in expectation, while the uniform-price assignment that retains its premium loses twice the least possible number.

Part (ii) shows that the obstacle lies in the incentives rather than in the veto alone. A mechanism that pays each investor the expected externality of his report (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979) gives him a net burden that depends on his own cost, through a payment for the flexibility he reports, and on the average of everyone's, but not on the realized imbalance: an investor whose cost is high bears about $1/(2ng_c(0))$, and an investor who is moved bears his cost less the payment for his flexibility, which is positive for those moved whose costs exceed about $0.37/(\sqrt{n}\, g_c(0))$, so that with truthful reports withdrawals stay bounded, near the level that binding enrollment with the leave-one-out rebate produces. That arrangement cannot be relied on, as part (iii) shows, and the reason can be put in one line. When participation binds, an investor chooses his report to minimize the expectation of his net burden, which is the objective that the expected-externality transfers align with everyone's, whereas under a veto he chooses it to maximize $\mathbb{E}[(v_i - b_i)^+]$, the expectation of the positive part of his margin over his burden b_i, so that the distribution of the burden and not only its mean governs the report, and a report that lowers the burden in the states he will accept and raises it in those he will refuse is attractive whatever it does to the mean. A claim of zero cost is such a report: the payment rewards a claimed willingness to be moved, and the veto lets an investor collect it when his favored project is not crowded and refuse the move when it is, so that the claim costs him nothing; the veto turns a payment for flexibility into an option, which is the Bayesian counterpart of Proposition 20, and the investors with a reason to exercise it are of order $\sqrt{n}$ in number and, having reported the lowest costs, would be the first moved. Whether some mechanism that is incentive compatible in the Bayesian sense under a retained veto keeps the shortfall bounded is open. The principle that the three parts share is that, at the moment an investor can still refuse, his net burden must not move with the common imbalance and must not rest on a claim that his refusal allows him to withdraw.

An expectation is a weak standard for a reserve. The condition that a mechanism be unsubsidized in expectation lets it pay subsidies in ordinary states and recover them in rare ones. A mechanism that pays every investor placed in the project that is not crowded without him the price he faces, and charges the others nothing, loses no one and runs a cash deficit of order $\sqrt{n}$ in every state, and if it adds a fixed charge on every other investor whenever one designated investor reports a cost below a threshold chosen so that the event has probability of order $n^{-1/2}$, the charge preserves dominance, since the charge in no investor's transfer depends on his own report, recovers the subsidies in expectation, and loses investors only on that event, so that the book closes without replacement capital with probability tending to one while $\mathbb{E}[Q_n]$ grows like $\sqrt{n}$, as part (i) requires. What such a mechanism needs is cash in every ordinary state, and the measure of what a veto costs is therefore the capacity a closing draws on in the state that arises, places and cash together, which is the standard that Section 3 applies to the escrow, funded in the state in which it pays. On that standard the reliability a bounded capacity can buy has an exact answer.

Proposition 26 (Reliability under a retained veto)**.** *Under the hypotheses of Proposition 25, let margins have a finite second moment, let $\lambda \in (0, \lambda_v)$, and for $\delta \in (0,1)$ let $\mathcal{S}_n(\delta)$ be the least s with $\mathbb{P}(Q_n + \lambda\Delta_n \le s) \ge 1-\delta$, the capacity, counted in places with cash converted at the rate λ, that suffices with probability $1-\delta$, which is the $S_n(\delta)$ of Corollary 23 when the mechanism needs no cash.*

(i) For every mechanism of the kind described before that proposition whose transfers make truthful reporting dominant in the assignment, with truthful reports, $\liminf_n \mathcal{S}_n(\delta)/\sqrt{n} > 0$ for every $\delta < \frac{1}{2}$.

(ii) Let each placed investor pay the price he faces, in absolute value, when placed in the project that is crowded without him, and receive that amount, whichever project he is placed in, when the number of the other investors who favor the project that is crowded without him is even. Under this mechanism truthful reporting is dominant in the assignment, $\mathbb{P}(Q_n = 0,\ \Delta_n \le s) \to \frac{1}{2}\mathbb{P}(Z^2 \le 2g_c(0)s)$ for every $s > 0$, so that $\limsup_n \mathcal{S}_n(\delta) < \infty$ for every $\delta > \frac{1}{2}$, and $\mathbb{E}[Q_n]/\sqrt{n} \to \gamma\,\mathbb{E}|Z|/4$.

Reliability above one half therefore costs capacity of order $\sqrt{n}$ under every mechanism in the class, and any reliability below one half can be had for a bounded amount, for a reason that is the payment identity once more. An investor's transfer may depend on every report but his own, so that the mechanism must guess, from the others' reports, the side on which each investor's own report will place him, and it is right about him in a state exactly when its guess matches that side; two states that differ only in one investor's favored project cannot both be guessed right, so that the states in which the mechanism is right about nearly everyone form a set no two of whose members are neighbors on the hypercube of favored projects, and such a set has probability at most one half. In every other state the mechanism is wrong about a positive share of its investors, each of whom either bears a quarter of the price, and refuses with a probability proportional to it, or is paid a quarter of the price in cash, and whichever a mechanism chooses, the sum over a share of n

investors of a quantity of order $1/\sqrt{n}$ is of order $\sqrt{n}$; charges that would finance the payments lose investors at a rate of at least κ_v, which the conversion rate λ records.

The mechanism of part (ii) is a device rather than a proposal, since it pays investors according to the parity of the others' favored projects, but it shows that the threshold of one half is sharp. In the states in which the number of investors favoring the crowded project is odd it guesses everyone right, nobody refuses, and its only cash outlay is the price paid to the investors moved, whose total converges to $Z^2/(2g_c(0))$; in the other states it is the uniform price whose premium goes to the uncrowded side. Its expected shortfall equals that of the leave-one-out rebate, so that the bound of Proposition 25(i) does not single out one mechanism, while the two distributions differ entirely: under the rebate $Q_n/\sqrt{n}$ converges to $\gamma|Z|/4$, so that no bounded reserve closes the book with any fixed probability, whereas under the parity device half of all books close on a bounded reserve and the other half need $\gamma|Z|\sqrt{n}/2$ places. The constant in part (i) is explicit in the proof but not sharp; what the proposition settles exactly is the reliability at which the capacity a veto requires passes from bounded to order $\sqrt{n}$. With participation binding before the imbalance is known and the premium returned within the book, every reliability below one is reached with a bounded number of invitations, by Theorem 22(iii) and Proposition 24; with a retained veto, no reliability above one half is. The proposition counts the capacity that an always-completed closing consumes, and an institution that holds only that capacity cancels when it is exhausted; cancellation then depends on reports, and the payment identity on which both lower bounds rest no longer characterizes incentive compatibility, as Theorem 22(iv) already shows, since under the uniform-price assignment with the premium retained the investor who sets the price gains by shading it. A lower bound for mechanisms whose finite reserves and cancellation rules are themselves part of the incentive problem is what separates these results from an implementation theorem, and it is open.

The information that does the work in Proposition 26 is the privacy of favored projects rather than of costs, and the parity device shows where the boundary lies.

Corollary 27 (Verifiable favored projects)**.** *Under the hypotheses of Theorem 22, with the withdrawal rule and the replacement capital of Proposition 25, let favored projects be verifiable, so that reports concern costs alone and the mechanism lies outside the class of that proposition, and let the D_n supporters of the crowded project with the lowest reported costs be moved, each paid the (D_n+1)-th smallest reported cost among those supporters, the other investors paying and receiving nothing. Truthful reporting is a dominant strategy, under a retained veto no investor withdraws, so that $Q_n = 0$, and, writing $X_{(l)}$ for the l-th smallest cost among the supporters of the crowded project, $\Delta_n = D_n X_{(D_n+1)}$ converges in distribution to $Z^2/(2g_c(0))$, with $\mathbb{E}[\Delta_n] \to 1/(2g_c(0))$ and, for costs uniform on $[0,1]$ and every even n, $\mathbb{E}[\Delta_n] = n/(2(n+1))$.*

With favored projects verifiable, the book is a procurement auction for the right to reassign investors, the price paid to the investors moved covers their costs, nobody who keeps his place pays anything, and the cash the book needs is bounded whatever veto its investors retain; the parity device attains exactly this outcome in the states in which the number favoring the crowded

project is odd, because parity tells it each investor's side there, and the first step of the proof of Proposition 26 shows that without verifiability no mechanism in its class can tell the sides of nearly all of its investors in more than about half of the states. The qualification is Proposition 20: a favored project that an investor declares rather than one that can be established is a claim, and paying those who leave the crowded project invites investors to claim it, so that the corollary applies to permissions fixed by something outside the book, a regulatory or tax constraint that a register records, and not to preferences an investor states, which a register can record but cannot verify. The capital requirement of a retained veto therefore arises from the interaction of three things, private preferences, truthful allocation and a retained right to refuse, and an institution that removes any one of them, by verifying mandates (Corollary 27), by dispensing with incentive compatibility (Proposition 25(ii)) or by binding participation before the imbalance is known (Theorem 22(iii)), escapes it.

A closing draws, in all, on capacity of four kinds, which differ in when they are called and in what becomes of them. Assurance against strategic refusal, the escrow of Section 3, is called in a profile of commitments that the mechanism eliminates, so that under known willingness it is never drawn and one promisor with disjoint groups of investors and an order of seniority can hold it for several investments at once (Proposition 8); insurance against genuine failure to enroll, which a book with private margins can suffer when too few investors are willing, is paid on the path of play and consumed, the same escrow serving as both, so that which cost it bears, its carry or its expected payout, depends on whether willingness is known (Section 7); replacement capital, which is principal, funds a place that an investor has refused and becomes invested; and the cash behind a mechanism's own payments must be available when the payments settle, which is the backstop of Table 4 in Appendix B.3. The first can be reused because its claims never arise, the third cannot because its claims are met with capital that stays invested, and whether the claims on the others arise in states exclusive enough for one fund to back several closings, or accumulate across them, is the part of the question that a model of successive closings would have to settle.

The expected-externality transfers have a use of their own under binding enrollment, where the veto that defeats them is absent: with those transfers and no fee, truthful reporting is a Bayesian equilibrium among the placed investors, their net burdens sum to C_n in every state, the threshold rule of Theorem 22(iii) is a best reply to every enrollment behavior of the others, and, if $S_n/n \to 0$, the shortfall converges to a Poisson variable with mean $\gamma/4$ (Corollary 42 in Appendix B.4).

Table 1 collects what the arrangements of this section buy. Binding participation removes the option to refuse a costly assignment; Bayesian incentives use beliefs about the other investors to balance the transfers, at the price of an equilibrium rather than a dominant strategy; dominant-strategy incentives give the stronger guarantee, and the leave-one-out rebate gives it with a bounded premium not returned. Neither constant under binding enrollment is claimed to be the least possible.

Replacement capital must itself be committed, and it is not free. If holding a unit of standby ready over the closing window costs ι_V and holding an investor's enrollment open costs ι_E, the replacement capital that a retained veto requires for reliability $1 - \delta$, with the premium retained,

Table 1: Reporting guarantees and shortfalls in the benchmark book

Arrangement	Reporting guarantee	Shortfall
Binding enrollment, premium returned in equal shares (Theorem 22)	Gain from a misreport of order $1/n$	Poisson, mean $\gamma/4$
Binding enrollment, leave-one-out rebate (Proposition 24)	Truthful reporting dominant	Poisson, mean $3\gamma/4$
Binding enrollment, expected-externality transfers (Corollary 42)	Bayesian equilibrium	Poisson, mean $\gamma/4$
Retained veto, favored projects private, any efficient mechanism with dominant-strategy incentives (Propositions 25, 26)	Truthful reporting dominant	Expected places plus λ_v times expected cash at least $\min\{2\kappa_v/g_v(0), 1\}\gamma\,\mathbb{E}\lvert Z\rvert\,\sqrt{n}/4$; capacity of order $\sqrt{n}$ for any reliability above one half
Retained veto, leave-one-out rebate with a fee	Truthful reporting dominant	$Q_n/\sqrt{n} \Rightarrow \gamma\lvert Z\rvert/4$; cash bounded
Retained veto, parity device	Truthful reporting dominant	Half of all books: no places, cash $\Rightarrow Z^2/(2g_c(0))$; the rest: $\gamma\lvert Z\rvert\,\sqrt{n}/2$ places and bounded cash
Retained veto, favored projects verifiable, movers paid (Corollary 27)	Truthful reporting dominant	No places; cash $\Rightarrow Z^2/(2g_c(0))$
Retained veto, expected-externality transfers	Not an equilibrium	Bounded under truthful reports, which the veto undoes

costs about $\iota_V z_{1-\delta/2}\gamma\sqrt{n}/2$, which grows without bound, while under binding enrollment with the premium returned it costs ι_E times a number that does not grow with the population, so that the comparison of orders survives unless ι_V/ι_E vanishes like $1/\sqrt{n}$. Restoring positive expenses of standing ready joins the theorem to Section 3. Under binding enrollment, the formation structure among the investors willing to enroll is a count, any n of them, and its commitment support R^* is the sum of the $n-1$ smallest of their expenses of standing ready, so that each additional willing invitee weakly lowers the escrow that robust formation requires; the escrow pays only when the book fails, while the replacement capital supplies principal when it closes. Binding enrollment resembles the blind pool of Proposition 15 in asking investors to give up a veto before information arrives, but the veto it removes is a veto over terms rather than over investments, since an enrolled investor obtains either a place in the project he favors, at a price, or a place in the other at a cost no larger than that price; the book keeps the selectivity that the pool gives up while acquiring the pool's indifference to the common shock.

Three qualifications bound the result. Both the funding and the permissions must bind, since a mandate attached to capital that its owner can still withhold is not capacity on which a closing can rely. The replacement capital of part (i) is principal, which funds a place that would otherwise be empty, and once it funds an illiquid investment it is deployed and cannot back the next closing. And the theorem treats two equal projects, a single closing, equal commitments and every investor flexible; its constants depend on that symmetry; the two departures considered next change the constant of part (iii) and, with unequal commitments, the scale of the results, but not their form, while for other departures, such as projects of unequal size or successive closings that draw on the same investors, the results are not established here. With a share φ of flexible investors who learn whether they are flexible only with their costs, a narrow mandate being an infinite cost, the argument of part (iii) applies on the event that the program can be financed, whose probability tends to one, the expected least cost converging to $1/(4\varphi g_c(0))$ by Theorem 19(iii), so that the Poisson mean becomes $\gamma/(4\varphi)$; investors who know their mandates when they enroll expect different burdens, as in part (v), and the constant rises. With unequal commitments q_i the imbalance in capital has standard deviation $\frac{1}{2}(\sum_i q_i^2)^{1/2}$, so that replacement capital as a share of committed capital scales like $1/\sqrt{n_{\text{eff}}}$, with $n_{\text{eff}} = (\sum_i q_i)^2/\sum_i q_i^2$, rather than like $1/\sqrt{n}$, a larger share when commitments are concentrated or accounts share a decision maker. Read together with Theorem 18, the theorem gives control two forms, each with a financing cost: an investor who keeps a narrow mandate is accommodated by flexible capital, of order $\sqrt{n}$ when mandates are idiosyncratic, and an investor who keeps the right to decide later is accommodated by replacement capital, of order $\sqrt{n}$ times a quantile of the common imbalance under a retained veto and of order one when the burden is accepted in advance and transfers are returned within the book.

4.5 When discretion should not absorb an imbalance

The preceding results treat differences in mandates as differences in fit. If investors avoid a project because they know something about its quality, flexible capital would systematically inherit

the investments that informed investors decline. The benchmark that follows measures this by expected value, as for risk-neutral investors. Let two projects be equally likely to be the good one, worth v^+ per unit, while the other is worth $v^- < 0$, with $v^+ + v^- > 0$, so that both are worth financing before anything is learned, and let each investor favor the good project with probability $\frac{1}{2} + \varepsilon$, independently given which it is.

Proposition 28 (The band of discretion)**.** *Given the imbalance $d = n_A - n_B$ of favored projects, financing the less favored project at its requirement has positive expected value if and only if*

$$|d| \;<\; \bar{d}(\varepsilon) = \frac{\log\left(v^+/(-v^-)\right)}{\log\left((1+2\varepsilon)/(1-2\varepsilon)\right)}.$$

The band $\bar{d}(\varepsilon)$ narrows as mandates become more informative and widens without bound as ε falls to zero. For fixed $\varepsilon > 0$, the probability that the imbalance lies within the band tends to zero as n grows, and the less favored project is the bad one with probability tending to one.

For a known precision the band is a fixed number of investors, so that in a large population almost every imbalance should be respected. When it is uncertain whether mandates carry information at all, imbalances of order $\sqrt{n}$ are consistent with independent differences in fit and imbalances of order n are not, so that the scale of an imbalance is evidence about its source, an inference that the proposition does not formalize; a common taste also produces imbalances of order n, and scale alone cannot tell a shared preference from shared information. The proposition takes declared mandates as given, which is natural when they are fixed at commitment and each declaration moves the imbalance by a single unit, and it identifies the limit of broad delegation: absorbing every imbalance, as a blind pool does, is valuable when mandates express fit and lowers expected value when they express knowledge. The criterion is the expected value of the less favored project taken alone, and for risk-averse investors it is not a welfare boundary, since a project with negative expected value can hedge the other: with a posterior of 0.9 that A is good, $v^+ = 2$ and $v^- = -1$, financing B has expected value -0.7, yet holding both projects pays 1 for certain, while holding A alone is worth about 0.93 to an investor with constant absolute risk aversion of one. A welfare criterion for risk-averse investors would value each project at investor-specific state prices.

Collective search. Finding the investors whose mandates admit a program, and assembling their commitments on compatible terms, bears on the three quantities that this section and the last have priced, the shortfall of principal B^*, the commitment support R^* and the discretion a book needs, and not always in the same direction (Proposition 43 in Appendix B.5). Adding willing investors, or widening permissions without making any investor unwilling, weakly lowers B^* and R^*, and lowering any investor's expense of standing ready weakly lowers R^*; search that leads a share of narrow mandates to follow a common recommendation raises the flexible share required to a positive limit and, when the recommendation shifts favored projects in the same way and costs satisfy the hypotheses of Theorem 19(iv), the cost of discretion per investor to a positive

limit; and better outside options that make some investors unwilling leave a formation structure contained in the original one, so that B^* and R^* weakly rise. A technology that searches on behalf of a population therefore cheapens commitment when it finds complementary permissions and makes discretion dear when it leads many investors to the same recommendation, and it bears on the replacement capital of Corollary 23 as well, since capital identified before the imbalance is known as willing to take a vacated place on stated terms is replacement capital. The third margin is the one through which better separate accounts act, and Section 6 takes it up.

5 What individual exposure costs

Section 4 showed that exposure need not be held in common for an investment to form, but individual exposure has costs of its own, and individual terms have further ones. The first is incentive pay, since pooled records can make delegated research much cheaper. The second is administration: forming the vehicles through which each investor holds his positions, recording those positions and their cash flows, enforcing the obligations that run between accounts, such as capital calls and escrowed claims, which a partnership agreement supplies to its partners and a book must supply by contract, and respecting each investor's tax, regulatory and legal circumstances, which determine which vehicles he may share with whom. Individual terms carry a further cost, specific to finance: equal-treatment clauses make terms common by extending a concession granted to one investor to everyone entitled to his terms. This section prices the three.

5.1 Delegation and pooled records

The most familiar reason for a fund is that investors delegate to a manager who knows more than they do, and that delegation, the argument runs, requires a mandate covering all of them at once. The argument fails when records are complete, and what survives of it is a cost of individual exposure rather than a requirement of common exposure. States of nature form a finite set Ω with a full-support prior p. A manager chooses an experiment e from a finite set E at cost $c_e \geq 0$, observes the cell of a partition $\mathcal{P}_e$ of Ω that contains the state, and then chooses a joint allocation $x = (x_i)_{i \in N}$ from a menu $\mathcal{M} \subseteq X$, where X is the finite set of feasible allocations of positions to accounts. The experiment that reveals nothing and costs nothing belongs to E. Account i's record is $r_i = (x_i, y_i)$, where its outcome $y_i = Y_i(x_i, \omega)$ depends on its own position and the state; public states of nature, where there are any, are recorded in every account. A *plan* π is a probability distribution over pairs (e, g) of an experiment and a rule $g : \mathcal{P}_e \to \mathcal{M}$; it has an expected cost c_π, and it induces a law μ_i^π of each account's record and a joint law μ^π of all records. The manager is risk neutral, his payoff is expected compensation less c_π, and any private benefit he derives from positions is absorbed into the cost of a plan. Compensation is *own-account* if it is a sum $\sum_i \phi_i(r_i)$ of payments, each paid by an investor and determined by his account's record, and *joint* if it is any function $\Psi(r_1, \ldots, r_n)$ of all records. A target plan $\tau = (e^*, f)$ is a pure plan whose allocations lie in $\mathcal{M}$. A plan *replicates* the target if it induces the target's record law in every account, $\mu_i^\pi = \mu_i^\tau$ for

every i, and compensation *strictly implements* the target if the target is optimal for the manager and every optimal plan replicates it.

Theorem 29 (Separate accounts see marginals)**.** *Consider plans available on $\mathcal{M}$.*

(i) Some own-account compensation makes the target optimal if and only if no plan that replicates the target costs strictly less.

(ii) Some own-account compensation strictly implements the target if and only if, in addition, every replicating plan that costs no more than the target assigns probability one to pure plans that themselves replicate it.

Both equivalences hold whether or not compensation is required to be nonnegative.

The theorem is the detectability characterization of moral hazard (Legros and Matthews, 1993; Fudenberg, Levine and Maskin, 1994; Rahman, 2012) applied to signals that are the accounts' marginal records: own-account compensation pays any plan $\sum_i \langle \phi_i, \mu_i^\pi \rangle$, so that by the theorem of the alternative the target fails to be optimal exactly when a mixture of deviating plans matches its record law account by account at a lower cost, and such a mixture is itself a plan the manager can follow. What separate accounts cannot see is the correlation between their records, and whether that matters depends on whether investors care about anything they cannot see, which with complete records they do not.

Corollary 30 (What owners value, own accounts can buy)**.** *Suppose records are complete, so that each investor's payoff from a plan depends only on the law of his own record and on what he pays, and suppose the conditions of Theorem 29(ii) hold for plans available on X. Then for every menu that contains the target's allocations, including the menu that permits every combination of the positions each investor allows in his own account, some own-account compensation strictly implements the target, and every plan that is optimal under it leaves every investor exactly as well off as the target.*

Joint control of a manager's actions is therefore not needed to make investigation his choice when outcomes are contractible account by account, although it can make that choice cheaper (Szalay, 2005; Dybvig, Farnsworth and Carpenter, 2010); Appendix B.6 works through an economy built to suggest the opposite, in which own-account pay of $5/4$ for each account that holds a program and earns its target return makes investigation the manager's strict choice, so that a joint mandate's one distinctive feature there is the cost of building it. Pooled records can lower that cost further, and paying a manager together is itself a formation problem; Appendix B.7 states both results, which this paragraph summarizes. Proposition 44 is the minimum-likelihood-ratio result for a risk-neutral agent with limited liability (Innes, 1990; Demougin and Fluet, 1998) applied to joint and marginal records: when a single deviation must be deterred, the least expected compensation is the saving the deviation offers the manager divided by one minus the smallest likelihood ratio between the deviation and the target, and the joint ratio never exceeds the smallest marginal one, so that own-account compensation costs more by the ratio of the two complements.

The difference is largest when information improves many bets a little. With n independent bets each held by a different account, as in co-investment programs and deal-by-deal vehicles, right with probability q_1 under investigation and $q_0 < q_1$ without it, own-account compensation pays $cq_1/(q_1 - q_0)$ whatever the number of accounts, while compensation paid only when every bet is right pays $c/[1 - (q_0/q_1)^n]$, which approaches c: with a five-point edge, $q_0 = 0.5$ and $q_1 = 0.55$, the first is $11c$ and the second, over 20 accounts, $1.17c$, a difference that decides whether investors who together value the investigation at $2c$ can afford it. The advantage belongs to pooling across bets rather than across investors, since an account holding an equal slice of every bet sees the number of right bets in its own profit and loss, which is the diversification argument of Diamond (1984). Figure 1 draws the two schedules. Proposition 45 then shows that when no single account's payment covers the cost of research, the investors' decisions to engage a manager form a formation game of the kind studied in Section 3, in which universal refusal is an equilibrium exactly when no investor alone values the target above the least payment that makes the manager follow it: in an economy with four concentrated accounts, each investor values investigation at 2 while a lone account would have to pay 6, and payment on the joint record lowers the least compensation to about 1.93. Pooled exposure thus economizes on incentive pay for investors whose accounts would otherwise be concentrated, and that saving, like the saving on records below, is a cost of individual exposure rather than a condition of committing together.

5.2 Exposure profiles and terms held in common

Pooled exposure economizes on records as well as on incentive pay, and the saving can be priced. Let a book hold m assets for n investors and offer k exposure profiles, each investor holding a combination of them through vehicles of the class that his tax status, regulatory regime or jurisdiction requires, at a cost κ for each record of an interest or of a position; a pooled fund is the case $k = 1$. For investors with constant absolute risk aversion and jointly normal returns and background wealth, the least aggregate certainty-equivalent loss from restricting investors to combinations of k profiles is, when combinations may be signed, half the sum of the squared singular values beyond the k-th of the matrix of their risk-adjusted optimal exposures, and no smaller when they must be held long: the theorem of Eckart and Young (1936) in the metric of the investors' losses, of which the mutual fund theorem is the case in which the second singular value is zero. In the dense benchmark in which every class holds the same profiles through vehicles of its own, the number of profiles a book should offer weakly rises as the cost of a record falls and weakly falls as the number of classes of investors who cannot share a vehicle rises, and heterogeneous background exposures can raise the rank of the optimal exposures to one more than the dimension of their span (Proposition 46 in Appendix B.8). A common profile costs most to the investors whose background wealth, liabilities or concentrated holdings set them apart from the population, since no profile shared with others can serve them, and cheaper vehicles, interests and position records move the book toward an exposure of each investor's own, while every class that needs vehicles of its own makes each profile dearer. Records and incentive pay are costs of

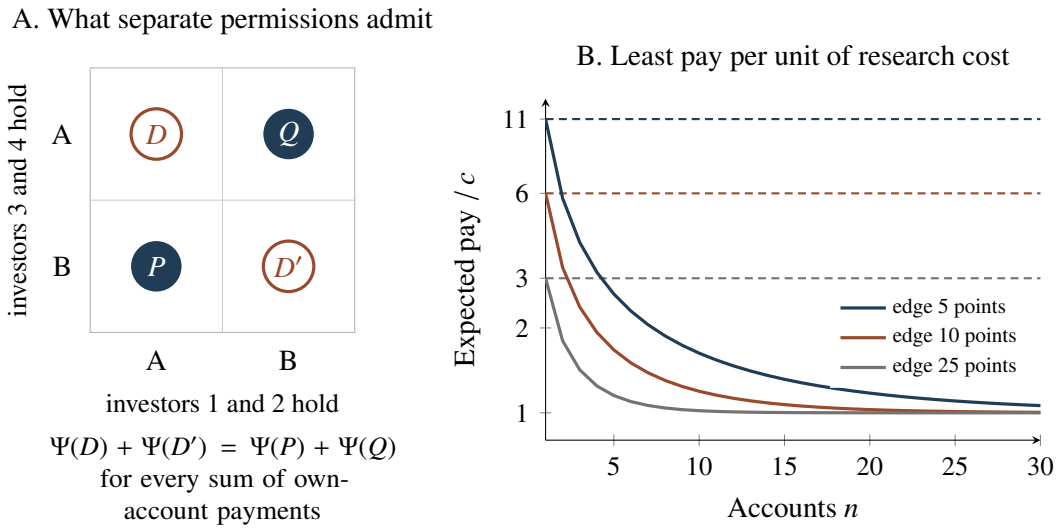


Figure 1: Two views of a fund. Panel A: in the four-investor economy of Appendix B.6, separate permissions that admit the targets P and Q also admit the recombinations D and D', but every sum of own-account payments is modular on these four allocations, so no collection of separate accounts can pay for P and Q while penalizing both recombinations; Corollary 30 shows that it need not. Panel B: the least expected compensation, per unit of research cost, that makes a manager investigate when each of n independent bets, each held in a different account, is right with probability 0.5 without research and 0.5 plus the edge with it. Dashed lines: own-account compensation on these concentrated accounts, which does not depend on n. Solid lines: compensation paid on the joint record, which approaches the research cost itself, and which an account holding a slice of every bet could also attain on its own record (Proposition 44).

individual exposure and nothing more: at a fixed willing set and fixed expenses of standing ready they leave the cost of commitment support unchanged, since by Proposition 17 exposure enters it only through who is willing, and they leave the discretion a book needs unchanged, since its amount is set by permissions (Theorem 18) and its cost by the costs of being moved (Theorem 19), which pro rata exposure raises.

Individual terms are constrained differently, by the need to obtain a collective agreement. The investors in a shared vehicle do not pay one another; what each receives is fixed by the program's terms, its fees, carried interest, allocations and rebates, and most-favored-nation clauses extend a concession granted to any one of them to every investor entitled to his terms, an entitlement that private equity funds grade by commitment size (Begenau and Siriwardane, 2024). Like the most-favored-customer clauses of Cooper (1986), equal treatment is a commitment that makes a selective concession expensive, and it bites at the moment the formation problem operates, because a concession needed to obtain a late, pivotal commitment is extended to investors who signed earlier. Securing a pivotal investor whose shortfall is z costs, under rate matching, z times the ratio of the commitments entitled to his terms to his own, and under final-terms matching a convex, piecewise-linear amount between z and that multiple, with a kink wherever the pivotal

investor's reduced charge passes an entitled investor's existing one, so that the cost of repair can exceed every available rent and existing fee tiers decide by what factor the two forms of matching differ (Proposition 47 in Appendix B.9). The entitled investors could waive their claims, but renegotiating terms held in common is itself a formation problem, in which each waiver is a contribution, refunded if the program fails, toward a deficit that is the threshold, so that universal refusal to waive is an equilibrium whenever no single waiver covers the deficit, while soliciting waivers one at a time removes that equilibrium, as an order does in Section 3.4, and the order of solicitation then fixes who gives up what (Proposition 48). Equal treatment raises the price of assurance by the same logic: a failure payment that every investor can claim must exceed, in some forming coalition, the second-largest expense among its members, and the liability it creates is at least R^* (Proposition 49 in Appendix B.9.2). Individual terms thus meet the constraint that individual commitments meet, a collective agreement, where individual exposure meets only the cost of records and incentive pay.

6 Outside options and the frontier

Commitment support and discretion make an investment form when its investors want it; they cannot make an investor commit to an investment he would rather not join, and the technology that cheapens separate accounts raises the alternative against which he decides. This section describes what an improvement in what an investor can do alone does to an investment whose only escrowed term is the conditional subscription, in the formation game of Section 2, and Corollary 33 then bounds what assurance can add to it. The results hold for two formation technologies: threshold programs and divisible contributions, both developed in Appendix B and summarized in Section 6.4 together with the cases in which each hypothesis fails.

6.1 Frontier equivalence

Definition 2 (Completion)**.** The game at o has *completion* if every individually acceptable profile is weakly Pareto-dominated by a pure equilibrium: for every $a \in \mathcal{A}(o)$ there is $a^* \in \mathcal{E}(o)$ with $g_i(a^*; o) \geq g_i(a; o)$ for every i.

Completion asks that whatever every investor would accept be weakly improved upon, for every investor, by some pure equilibrium. It is a property of the game's payoffs and presumes neither an algorithm nor a selection rule. Appendices B.10 and B.13 show that it holds for both technologies.

Theorem 31 (Frontier equivalence)**.** *Suppose Assumptions 1 and 2 hold and the game at o has completion. Then:*

(i) the equilibrium and acceptable frontiers coincide: a vector of $g(\mathcal{A}(o); o)$ is undominated in that set if and only if it belongs to $g(\mathcal{E}(o); o)$ and is undominated there;

(ii) the selection gap equals the largest aggregate surplus among acceptable profiles, so that $H(o) = W^*(o)$ and the instability term in (6) vanishes:

$$G(o) = \sup_{a \in \mathcal{A}(o)} \sum_i g_i(a; o); \tag{12}$$

(iii) every acceptable profile that maximizes aggregate surplus yields the payoff vector of some pure equilibrium.

Proof. By Lemma 2, $G(o) = \sup_{a \in \mathcal{E}(o)} \sum_i g_i(a; o)$, which is at most the right side of (12) because equilibria are acceptable. Completion maps each acceptable a to an equilibrium with weakly larger incremental payoffs and hence weakly larger aggregate surplus, which gives the reverse inequality. If a maximizes aggregate surplus over $\mathcal{A}(o)$, the equilibrium that dominates it cannot have larger aggregate surplus, so the two payoff vectors are equal. Part (i) follows from the same domination and the nesting (5); Appendix A gives the details. □

Part (iii) bounds what coordination under fixed terms can achieve. A coordinator who computes the best individually acceptable allocation has found an outcome the investors could already reach as an equilibrium. Computing that allocation and leading investors to it can be valuable, and neither is free, but under unchanged terms and an unchanged set of potential participants neither can improve on the best equilibrium. Any further gain must come from changing what investors receive, through transfers, new terms, contingent payments or funding, or from enlarging the set of investors who can take part, which Appendix B.16 shows raises the best equilibrium without moving the floor.

6.2 Compression

Definition 3 (Contraction)**.** A change from o to o' *contracts acceptable payoffs* if every incremental payoff vector acceptable at o' is weakly dominated by one acceptable at o: for every $a' \in \mathcal{A}(o')$ there is $a \in \mathcal{A}(o)$ with $g(a; o) \geq g(a'; o')$.

A sufficient condition is a pointwise decline, $g_i(a; o') \leq g_i(a; o)$ for every profile and investor, because every profile acceptable after the change is then acceptable before it with weakly larger payoffs. An outside-only improvement satisfies this condition. If investor k's outside value rises by Δ while program payoffs are unchanged, his incremental payoff from every formed profile in which he participates falls by Δ, his payoff from a failed subscription is unchanged relative to his outside value, and nothing else changes.

Theorem 32 (Compression)**.** *Suppose Assumptions 1 and 2 hold at o and at $o' = o + \delta$ with $\delta \geq 0$, the game at o has completion, and the change contracts acceptable payoffs. Let $\Delta = \sum_i \delta_i$. Then*

$$L(o') = L(o) + \Delta, \qquad G(o') \leq G(o), \qquad \Delta - G(o) \;\leq\; H(o') - H(o) \;\leq\; \Delta. \tag{13}$$

Both bounds on $H(o') - H(o)$ are attained.

Proof. Lemma 2 gives the first equality. For the second,

$$G(o') = \sup_{a \in \mathcal{E}(o')} \sum_i g_i(a; o') \;\le\; \sup_{a \in \mathcal{A}(o')} \sum_i g_i(a; o') \;\le\; \sup_{a \in \mathcal{A}(o)} \sum_i g_i(a; o) = G(o),$$

where the first inequality uses Lemma 2 at o', the second uses contraction, and the equality is Theorem 31. Hence $H(o') = L(o') + G(o') \le H(o) + \Delta$, while $H(o') \ge L(o') = H(o) - G(o) + \Delta$.

For attainment of the lower bound, let two investors form a program that requires both, with formed payoffs $(0.5, 10)$, outside values zero and no failure costs. Raising the first outside value to 0.6 destroys formation, and the best equilibrium falls from 10.5 to 0.6, a change of exactly $\Delta - G$. For the upper bound, let two investors form a program that requires both and pays $(8, 8)$, and add a third investor whose outside value of 10 exceeds the payoff of 1 that membership would give him; raising his outside value by Δ leaves the program intact and raises the best equilibrium by exactly Δ. □

The lower bound uses only Lemma 2; completion and contraction are needed for the bound on the range and for the upper bound, and Appendix B.14 shows how the bounds degrade when completion fails (Proposition 55). Under the hypotheses of Theorem 32, three cases arise when a single investor's outside option improves. If he lies outside the best equilibrium's membership, the best equilibrium rises by the full Δ. If he is a member of the best equilibrium whose slack is at least Δ, that profile remains an equilibrium with unchanged payoffs, so the best equilibrium cannot fall; in the threshold model of Appendix B.10 and the continuous model of Appendix B.13 it is unchanged, and the selection gap closes by Δ. If Δ exceeds his slack, the best equilibrium can fall by as much as $G - \Delta$. Because (13) compares best with best and worst with worst, neither bound depends on a selection rule.

6.3 The value of assurance

Theorems 31 and 32 also bound what an escrow can recover. In a game of binary commitments, a member who leaves a formed profile obtains his outside value whether or not his departure causes failure, and an outsider who joins leaves it formed, so payments to committed investors upon failure change no incentive at a formed profile.

Corollary 33 (The value of assurance)**.** *In a game of binary commitments satisfying Assumptions 1 and 2, in which an additional commitment never causes a formed investment to fail, escrowed payments leave the set of formed pure equilibria unchanged, so a schedule of failure payments that implements formation raises worst-case welfare from $L(o)$ to at most $H(o)$, by at most the selection gap $G(o)$, before the carry on its reserve. Under the hypotheses of Theorem 32, an outside improvement raises the floor by Δ and weakly lowers the most that assurance can add.*

An escrow is therefore worth least exactly where better separate accounts have compressed the range above the floor, and where compression has removed every formed equilibrium it can

recover nothing at all, since what remains is distribution gap that only new terms can close; Appendix B.15 shows how an improvement in one investor's outside option converts selection surplus into distribution gap at a given allocation.

6.4 Threshold programs, divisible contributions and collective search

The hypotheses of Theorem 32 hold in the two specializations that the rest of the paper uses, and fail under congestion. In a threshold program, in which the program forms once ν investors subscribe and an additional member weakly improves every existing member's net payoff, a property Appendix B.10 calls inclusion monotonicity, every individually acceptable profile is weakly Pareto-dominated by a pure equilibrium, the willing formed memberships are closed under union, their largest element is the best equilibrium, and it cannot grow as outside values rise (Proposition 50 in Appendix B.10); the willing-membership structure is the one that underlies Moulin mechanisms for cross-monotonic cost sharing (Moulin, 1999; Moulin and Shenker, 2001; Juarez, 2013), and it locates the best equilibrium by deleting unwilling members until the remaining set is willing. Completion holds as well when indispensable investors supply divisible capital to a program of minimum scale, without a lattice of equilibria or strategic complementarity (Theorem 52 in Appendix B.13), while congestion, which arises whenever capacity is scarce, can reverse the theorem (Proposition 54 in Appendix B.14); the theorem's natural habitat is therefore the first close, the club transaction and any program with a minimum committed amount, where scale or fixed formation costs bind, and where participation is congested the theory does not sign the effect. Two economies in Appendix B show the mechanism. In a five-investor threshold program whose outside values rise with a common technological improvement while its terms stay fixed, the best equilibrium is constant until a member leaves and falls discontinuously when he does, the floor rises linearly, and the selection gap closes long before the program ceases to create surplus, after which the entire value of coordination is distribution gap, part of which named-member contracts with balanced transfers recover (Figure 2 in Appendix B.10). In a four-investor economy with divisible capital, a fall from 10 to 7 in the per-unit cost of investing separately lowers best-equilibrium welfare from \$76,000 to \$22,000 although the shared program, which still needs the capital of the one investor whose separate investment has become attractive, would continue to produce \$76,000: he declines because proportional cost sharing would pay him less than he now earns alone, and the selection term falls from \$60,000 to zero while the distribution term rises from zero to \$54,000 (Appendix B.11.1).

Definition 1 separates two costs that technology lowers. One is the cost of an own term, of reading a mandate, keeping an account's attributable record and paying a manager on it; when it falls, outside values rise and records become complete, so that by Corollary 30 delegation can be arranged account by account. The other is the cost of a joint term, of the search, verification, escrow and joint-record pay that only a collective arrangement performs; when it falls, joint terms can do more. The first works on the floor of the equilibrium range and the second on its ceiling, and a technology that lowers both leaves an investor more or less willing according to where its

savings land relative to his slack, the incidence that Appendix B.17 formalizes. Collective search is a cost of the second kind, and it changes who can take part. Under inclusion monotonicity, a larger pool of investors that a program can reach leaves the floor where it is, weakly raises the best equilibrium, and raises it and the selection gap by exactly the gain in the best acceptable surplus (Proposition 57 in Appendix B.16), so that search and better outside options are opposite forces on the same object, the one enlarging the set of investors from which willing memberships are drawn and the other shrinking the family of such memberships. This is why, by Theorem 31(iii), search rather than computation is the margin on which coordination at fixed terms can raise what is attainable: a coordinator who computes the best allocation for a given pool can do no better than its best equilibrium, while one who finds the investors who belong in the pool can. In the five-investor economy, reaching one more investor whose alternative is no better than the cheapest member's keeps the program formed over a range of outside improvements in which the original pool has already failed (Figure 5 in Appendix B.16). Where capacity is scarce an entrant can harm incumbents, so that enlarging the pool can lower the best equilibrium, and a search technology that also raises outside values, by helping each investor find separate investments, combines both forces of this section.

7 Institutions and measurement

This section locates the commitment support of Section 3 and its substitutes in the institutions that assemble shared investments, analyzes the sensitivity of its cost to the assumptions that determine it, and sets out what measurement would establish, including the comparison between broad delegation and individually permitted books on which the institutional implication of Section 4 rests.

7.1 Where markets supply commitment support

Every instrument of Section 3 has a counterpart in practice, and Table 2 collects them. A venture round is led by an investor whose term sheet precedes the others' commitments, but almost all term sheets are expressly nonbinding apart from provisions on exclusivity, confidentiality and expenses (Wilson Sonsini Goodrich & Rosati, n.d.b), so that the lead becomes bound only when the purchase agreement is executed, usually at a closing alongside the others, which is conditioned on a minimum number of shares being sold (Wilson Sonsini Goodrich & Rosati, n.d.a), a condition that operates as a refund. The term sheet is therefore not the binding first move that an order requires. It is a promise made credible by the lead's reputation, which can deter reneging when the future business at stake exceeds the gain from it (Levin, 2003) but supplies no funds if the lead walks away; a lead who funds a first closing before others commit is an order in the model's sense, while at a closing at which all commit together the equilibrium of refusal is removed without payments only if one investor can complete the round alone (Theorem 3(iii)), so that a lead who takes nearly all of a round leaves little to coordinate, and one who takes a small part leaves the

closing exposed to the coordination failure of Theorem 3; subsequent closings on the same terms, within a window of 45 to 90 days (Gunderson Dettmer, n.d.), admit investors to a round that has already formed and are not rounds in the sense of Section 3.4, whose rounds all precede formation. A private fund admits its investors at a first closing, conditioned in many funds on a minimum, and at later closings, typically within twelve months of the first (Carta, n.d.); the minimum is a refund, which by Theorem 3 does not by itself remove the equilibrium of refusal. Subscription documents are signed one at a time over months and the running total is reported to the investors still deciding, but the order of signatures is an order in the model's sense only if each signed subscription binds its investor, conditionally on the closing, before the next is solicited; where subscriptions may be withdrawn until the general partner accepts them at a closing, the investors admitted at the first closing commit together, and later closings admit investors to a fund that has already formed, so that a first closing rests on investors able to close the fund alone, on reputation, or on subscriptions made irrevocable when signed. Organizational expenses, whose median caps in 2019–2025 were 20 to 25 basis points of commitments (Institutional Limited Partners Association, 2026), are charged upon formation; they would serve as the premium of Theorem 13(ii) if paired with payments upon failure above investors' expenses, which fund documents do not typically provide. A special-purpose acquisition company forms its first vehicle by a firm-commitment underwriting, an anchor whose spread is 5.5 percent of proceeds, 3.5 percentage points of it deferred and payable only if a merger is completed (Gahng, Ritter and Zhang, 2023; Klausner, Ohlrogge and Ruan, 2022), and holds the proceeds in a trust that returns them if no merger occurs, which is a refund; its sponsor's capital at risk, a median of $6.8 million, is an expense of standing ready that the sponsor bears alone (Gahng, Ritter and Zhang, 2023).

The record of enforcement shows that the expenses of failed attempts are material and that who bears them is contested: one adviser allocated $338 million of broken-deal expenses incurred between 2006 and 2011 to its flagship funds without attributing any part to co-investors, and the Securities and Exchange Commission found that $17.4 million should have been borne by them (Securities and Exchange Commission, 2015); in a take-private signed in August 2017, investors who had committed before signing objected to bearing the fees of the bridge facility that assured funding for the investors solicited afterwards, on the ground that their own firm commitments made it unnecessary for them (Securities and Exchange Commission, 2022). The bridge was an anchor for the later round, and the dispute was over who should pay for it.

7.2 The cost of commitment support and the puzzle of cheap capacity

Four quantities must be kept apart: the capacity a promisor must hold, R^*, or $\hat{R}_\ell$ with a staged escrow over ℓ public rounds; the carry of holding it over the closing window; the expected payments it makes when willingness is private and claims arise on the path of play; and the broader service of formation, which includes finding opportunities and investors, diligence, negotiation and administration, none of which the calculation measures. Appendix B.19 computes the first two under the paper's benchmark, and the paragraph that follows computes the third, for a fund of $300

Table 2: Contingent capacity and its substitutes in practice

Institution	Term	Instrument (result)
Venture round	Nonbinding lead term sheet; all sign at a closing with a minimum-shares condition	Reputation, not an order; anchor if the lead can complete the round alone (Thm. 3(iii)); refund
Private fund	Subscriptions signed and reported; first closing, often at a minimum; later closings	Order only if signed subscriptions bind before the first closing (Prop. 9); refund (Thm. 3)
Co-investment	Flagship fund bears expenses of deals that fail	Allocation of expenses of standing ready
Take-private with bridge	Bridge facility funds the purchase before later investors invest at closing	Anchor for a later round (Thm. 3(iii))
SPAC offering	Firm-commitment underwriting, part of spread contingent on merger; trust	Anchor; refund
SPAC combination	Redemption right exercised after the target and its terms are known; forward-purchase agreements and PIPEs signed with the merger agreement; non-redemption agreements signed later by particular holders	Retained veto (Thm. 22(i)); replacement capital committed before redemptions are known; binding tickets for the holders who sign
Rights offering with standby underwriter	Underwriter takes the shares not subscribed, for a standby fee paid whether or not it is called	Replacement capital supplied from outside the book, with an availability fee (Thm. 22(i), Cor. 23)
Restructuring backstop	Backstop parties fund any unsubscribed part of a plan's rights offering, for a backstop premium	Replacement capital for the whole offering, with an availability fee
Contingent offering	All-or-none or minimum sale with escrow of proceeds	Refund (Thm. 3)
Sponsor platform	One sponsor backs many vehicles	Portfolio of promises (Prop. 8)

Sources: Gunderson Dettmer (n.d.); Wilson Sonsini Goodrich & Rosati (n.d.a,n); Carta (n.d.); Institutional Limited Partners Association (2026); Gahng, Ritter and Zhang (2023); Klausner, Ohlrogge and Ruan (2022); Securities and Exchange Commission (2015, 2022, n.d.a,n). A refund returns committed money, or spares investors their expenses, if the investment fails and pays nothing above those expenses, so by Theorem 3 it does not by itself remove the equilibrium in which nobody commits.

million raised from 30 investors of equal size, with expenses of standing ready of one percent of each commitment, an input that is an assumption rather than an estimate, and with capacity costing either 50 basis points a year, the order of the commitment fees on undrawn credit lines that Berg, Saunders and Steffen (2016) show to be a principal component of the cost of committed credit, or 10 percent, the spread between a sponsor's cost of capital and the Treasury-bill yield when the sponsor holds the reserve itself (Table 8 in Appendix B.19). Under the known-willingness benchmark, the capacity that eliminates strategic nonformation when every investor is required comes to about one percent of the fund, and carrying it for a year costs half a basis point if a bank supplies it and about ten if a sponsor holds it, against organizational expenses capped at 20 to 25 basis points and management fees whose present value is $10 to $15 per $100 committed (Institutional Limited Partners Association, 2026; Metrick and Yasuda, 2010). That is a statement about one narrow function, which motivates measurement without establishing what the remaining charges buy, and whether any observed charge is the rent that Corollary 6 would let a promisor with capacity marginally above R^* collect is an empirical question about competition among promisors that the model does not answer.

The calculation raises a puzzle, for if contingent capacity is this cheap, escrows that pay investors more than their expenses upon a failed closing should be common, and they are not: practice relies on leads whose commitments are informal, on orders where earlier commitments bind, on refunds, and, in public offerings, on anchors. Three explanations fit the pattern, two supplied by the theory and one outside it. Under backward iterated dominance an order costs nothing (Proposition 9), so investors who have reason to believe in one another's future rationality, as sponsors and investors who meet repeatedly do, need no capacity at all, and the escrow becomes necessary only where commitments must be simultaneous or the sequence cannot be relied upon. Under private willingness, moreover, an escrow above expenses attracts investors who would rather the investment failed and screens them out, when unwilling investors can be arbitrarily close to indifference, only when it is paired with a charge (Theorem 13), whereas an order with refunds no larger than expenses screens without one (Theorem 13(iii)). The difference is large in the fund of Table 8: if each of its 30 investors is willing with probability 0.99, a simultaneous screening escrow is drawn whenever some investor proves unwilling, with expected payments of about $730,000 per attempt, 24 basis points of the fund, against expected wasted expenses of about $750,000; an order in which commitments are made one at a time wastes about $360,000, 12 basis points, and needs no payment at all once each investor's gain from the fund exceeds about $34,000, a third of his expense. Capacity that is never drawn when willingness is known is drawn on the path of play when it is private, and its cost is then the expected waste it insures rather than its carry. The combination observed in private funds, subscriptions collected one at a time with refunds, is therefore the one the theory selects where commitments can be sequenced and each binds when made, and a large lead plays the anchor's part in a venture round.

The third explanation is reputation: a lead whose term sheet does not bind is believed because reneging would cost it future business, which makes the promise credible through an incentive constraint rather than through capacity to pay, and the limit of such promises is the future surplus

of the relationship (Levin, 2003). Whether explicit, recorded commitments can replace the future rents that make informal ones credible, and whether search that lowers intermediaries' rents also lowers the franchise value on which informal commitments rest, are questions for further work. Where commitments cannot be sequenced, as in a public offering whose investors subscribe at once, the theory predicts contingent capacity or an anchor, and the firm-commitment underwriting is an anchor. The cost of time does not resolve the puzzle: among calendars implemented by staged escrows, Proposition 12 shows that a calendar beats capacity only if a round of delay costs less than the carry of one investor's expense over the round, which in that fund is about $42 for a month at 50 basis points a year and $833 at 10 percent; a month's delay in forming a $300 million fund almost certainly costs more, so the reliance of fundraising on orders indicates either that the time sequencing takes is consumed by diligence and negotiation in any case, or that credible capacity is scarcer than its carry suggests, a friction the model does not contain. Transactions that face deadlines, by contrast, rely on bridge facilities and firm commitments, which are anchors, as the take-private above did. The escrow is therefore, in the institutions examined here, a design proposal rather than a description, and so is the book of Section 4, with individually permitted accounts, discretion granted by many investors and priced at the crowded project, commitments that bind before the common imbalance is known, shared contingent support and individual exposure; whether either performs as the theory says is a question for the measurement that follows.

Between the fund and the book the theory draws a line of its own at the composition of mandates, as Section 4 set out: the book where they differ idiosyncratically, and the fund where a common component moves them together and the economies of common exposure can decide.

The costs this paper prices are borne by someone, and in an equilibrium in which investors supply contingent capacity at a price they should appear in required returns at formation. An investment that cannot exist without capacity R^* behind its commitments and, under a retained veto, replacement capital sized to a quantile of its shortfall must pay the carry of that capacity over its closing window, and the discretion its flexible capital grants has the bounded expected cost of Theorem 19, so that formation cost enters an investment's hurdle rate as the cost of trading enters a security's expected return in Amihud and Mendelson (1986). The paper does not derive that premium, which would require a market in contingent capacity that it does not model, and private-market returns are measured too coarsely to identify it across investments; the prediction it does support concerns selection rather than price. At a given expected return, the investments that form are those whose formation costs are smallest relative to the surplus they create, so that compression (Theorem 32), cheaper standby and earlier binding should change the composition of what forms before they change what it pays, and a test of the theory on returns should look for that change in composition first.

7.3 A cross-section written into contracts

The contracts of special-purpose acquisition companies contain the model's objects. In the paper's terms such a company is an anchor with a refund attached: its offering is typically underwritten on a firm-commitment basis, so that the underwriter forms it alone, and its trust account is an escrowed term in the sense of Definition 1, returning the offering proceeds with interest if no combination is completed, which makes the subscription credible without paying anyone for the failure of others; a trust funded above the offering price would add such a payment. Its sponsor is paid principally through shares that have value only if a combination forms. The redemption right is close to an outside option available only upon redemption at trust value, departing from Assumption 1 mainly because redeeming holders typically keep warrants whose value depends on closing; the minimum cash condition is a formation threshold; sponsor forfeitures and non-redemption incentives are transfers; and amendments and waivers change terms. Market data show these instruments moving together as interest rates rose and growth valuations fell in 2022. Barlow et al. (2023) report that redemptions averaged more than 80 percent of trust capital for deals closed in 2022 and that in March 2023 investors redeemed 95 percent on average. In their data, 94 percent of merger agreements signed in 2021 included a minimum cash condition, while among deals closed in 2022 only 77 percent required one to be satisfied; 60 percent of 2022 merger agreements were amended, against 37 percent in 2021; and sponsors forfeited founder shares in 85 percent of 2022 deals, against 56 percent in 2021. Sponsor forfeitures fund concessions from outside the class of public holders, and non-redemption agreements pay them to specified holders rather than to the class. The first is the first of the routes set out in Appendix B.9.3 and the second is selective retention outside any clause, but selective retention funded from outside the class is cheapest whether or not equal treatment binds, so establishing that it binds requires the entitlement provisions of each deal. The costs of formation fall on shareholders who do not redeem: for mergers between January 2019 and June 2020, net cash per $10 share averaged $4.10, with a median of $5.70 (Klausner, Ohlrogge and Ruan, 2022); Gahng, Ritter and Zhang (2023) document the full life cycle.

The theory makes four predictions for this cross-section. First, amendments, waivers and transfers should concentrate in deals whose minimum cash condition binds, relative to otherwise similar deals in which committed capital already clears it. Second, transfers from sponsors should be largest where the marginal holder's slack is smallest, because they address the distribution gap. Third, success-contingent incentives should change who remains without removing a deal's dependence on coordinated redemption decisions; only commitments that pay or fund whether or not others redeem remove that dependence. The redemption right is also the retained veto of Theorem 22: every public holder of a deal decides after the same target and the same terms are announced, so that redemptions should fluctuate across deals on the scale of their mean rather than concentrate, and the capital a sponsor must arrange to close should scale with a quantile of that distribution rather than with its average, which is what the forward-purchase agreements, and the PIPEs signed with the merger agreement, supply, replacement capital committed before redemptions are known, while non-redemption agreements bind the participation of the holders

who sign them. Fourth, therefore, the replacement capital a sponsor arranges, and what it costs, should scale with a high quantile of the cohort's distribution of redemptions rather than with its mean.

7.4 Measurement

The central empirical problem is to separate two reasons for declining to commit that look alike in transaction data: an investor may decline because his outside route improved, which is the mechanism of Theorem 32, or because the shared investment became less attractive, which is not. A persuasive design identifies changes that cheapen separate implementation, measures their effect on the shared route as well, and asks whether formation failures concentrate in cohorts with little participation slack or little headroom between the capital investors would accept and the threshold, and, by Proposition 47, where the cost of retaining the pivotal investor is high relative to the rent available for concessions, which requires measuring the commitments entitled to his terms and their existing terms rather than his size alone. Private markets lack the standardized contracts of special-purpose acquisition companies, so a test there must be prospective: it should classify each opportunity's constraints before outcomes are known, retain every considered opportunity, including refusals and failed closings, and record for each decision date the outside route then available, the requested allocation, net participation terms, authority, existing obligations and funding state, distinguishing refusal from diligence, documentation, consent, funding and settlement as reasons for failure. By Proposition 57, reductions in the cost of locating and admitting co-investors should raise the best attainable outcome for programs with binding minimum scale at unchanged outside options, while mattering less, or lowering outcomes, where membership is congested.

The theory identifies the quantities to measure, and they differ in how they can be observed. Some can be recorded directly. They are the number of independent decision makers, rather than of accounts, since a thousand limited partners bound by one fund's mandate make one decision and accounts controlled by one committee fail or defer together; the permissions each investor has granted and their overlap; the sizes and dates of commitments; and three attributes of each commitment at each closing. Its authority runs from an expression of interest, through authority to commit within stated limits, to an enforceable obligation with its remaining conditions; its availability is whether the cash can be delivered by the closing date, given its notice period; and its encumbrance is whether the same capital is already reserved, called, invested or promised elsewhere. Capital counts toward a closing only when it is authorized, eligible under its permissions, available on time and unencumbered, so that capital already invested does not count, and Theorem 22 implies that the replacement capital a book needs depends on the moment at which its investors' participation becomes binding. Other quantities must be inferred: the common component of permissions, and above all the distribution near zero of participation margins and of costs of being moved, which no record shows and which hypothetical questions do not reveal reliably, since answers depend on framing, on the alternatives in view and on whether they bind. They can be identified from

priced choices that bind, such as acceptances of standing offers whose terms are varied across comparable books under a design fixed in advance. A test should also distinguish access to an investment that would have been financed anyway from financing that would not otherwise have occurred, since only the second enlarges the set of investments that exist.

The implication of Section 4 is testable by running the same chronological sequence of opportunities through four arrangements: broad pooled discretion; a fund that individualizes through excuse rights, side letters and co-investment, which Section 4 reads as a partial book whose extent the cost of records limits; binding narrow mandates without discretion; and binding narrow mandates together with discretion granted by many investors and priced at the crowded project. Committed capital, underwriting standards, issuers' admission decisions, fees and the timing of information should be held fixed across the four, and permissions must be specified before the imbalance at each closing is known, since a retrospective optimization that adjusted mandates to the opportunities realized would exaggerate the benefit. That comparison holds committed capital fixed and therefore measures the efficiency of allocation. Whether an arrangement creates more executable financing is a separate question, which requires holding fixed the population of potential investors and the pipeline of opportunities, treating commitments as an outcome, and recording projected outcomes before each closing. The measures follow from the theory. The first comparison measures the fit between each investor's exposure and his objectives, capital left unused, the price paid for places in crowded projects, the cost of administering narrow commitments and their records, and the quality of the residual allocations absorbed by flexible capital relative to the rest, since Proposition 28 predicts adverse selection in them when mandates carry information. The second measures the reliability of formation, the delay in closing, the replacement capital reserved and the time for which it is reserved, and the commitment support each arrangement required. The theory predicts that the fourth arrangement preserves most of the reliability of broad delegation while improving the fit of exposure where differences in mandates are idiosyncratic, and not where they are common or informed.

8 Scope and conclusion

Each result holds in its own environment, and the link that Theorem 22 draws between the two parts of the paper is established in one particular environment. The results of Section 3 concern a binary commitment game with known gains, an expense of standing ready that does not depend on which coalition fails, and payment schemes contingent on the profile of commitments; mechanisms with richer messages, such as those that implement virtually in iteratively undominated strategies with small fines (Abreu and Matsushima, 1992), randomized or private orders, dynamic mechanisms robust to beliefs (Penta, 2015) and information design (Morris, Oyama and Takahashi, 2024) lie outside them, and Halac, Lipnowski and Rappoport (2021) show that uncertainty about one's rank in a sequence can lower the cost of unique implementation. Theorem 4 holds for every formation structure and any increasing utility, while Theorem 5 gives R^* for counts and the floor R_c in general,

for risk-neutral investors and for investors with strictly increasing concave utilities alike, and characterizing the requirement under the weaker standard for every structure is open; the example in which it falls below R^* relies on risk neutrality. The results on time rely on common belief in future rationality and do not hold under normal-form rationalizability, and Proposition 12 compares calendars implemented by staged escrows. Theorem 13 and Propositions 34 and 15 assume indispensable investors and independent willingness, which rules out informational cascades, and state their monetary conditions for risk-neutral investors, with the conditions under constant absolute risk aversion beside Proposition 15 and, for Theorem 13, in Appendix B.1; Proposition 8 assumes disjoint groups of investors and known willingness, so that no claim arises on the path of play.

The results of Section 4 take permissions as fixed before imbalances are known, place the book at critical load, treat units as interchangeable within a project and abstract from time: flexible principal placed in an illiquid investment cannot be reused at the next closing, and a dynamic model must track uncalled commitments, deployed capital, follow-on obligations and distributions. Theorem 19 assumes costs of being moved that are linear in the fraction moved and independent across investors, its bounded expected cost requires a positive share of flexible investors whose costs of being moved have a positive density at zero, and the budget-balanced variant discussed after it is incentive compatible only approximately. Theorem 22 assumes two equal projects, every investor flexible, margins independent of favored projects and costs, risk-neutral enrollment decisions and replacement capital that funds any vacated place on its terms, and, under binding enrollment, investors who know their margins but not their favored projects and costs when they enroll, except in part (v), which also lets them know their favored projects and costs, at critical load, with the premium returned, and among symmetric equilibria in enrollment rules that treat the projects alike. With the premium returned in equal shares truthful reporting is only approximately dominant, and Proposition 24 restores dominance with a rebate that leaves out each investor's own report, at the cost of a larger but still bounded shortfall. Propositions 25 and 26 concern mechanisms that make the efficient assignment with dominant-strategy incentives under a retained veto, with replacement capital that fills every vacancy without transfers, and they count the capacity a closing draws on as places and cash together; the first bounds its expectation from below, with a positive bound when margins have a finite mean, and shows that the expected-externality mechanism is not incentive compatible under the veto, and the second shows that, when margins have a finite second moment, no such mechanism closes with reliability above one half on bounded capacity, with a constant that is explicit but not sharp; Corollary 27 shows that both bounds rest on the privacy of favored projects, since with favored projects verifiable and reports confined to costs a procurement auction needs no places and bounded cash. Whether a mechanism that is incentive compatible in the Bayesian sense under a retained veto can keep the shortfall bounded, what reliability a given capacity buys below one half, and how the capacity behind a mechanism's payments and the capital behind its places are best held together over successive closings, are open, as is any bound for mechanisms that cancel when a finite reserve is exhausted, whose cancellation rules then enter the incentive problem. The bounds of Propositions 25 and 26 assume the efficient

assignment, and the question with the most at stake is how much real misallocation buys how much capacity: for a loss budget $\varkappa$, let $\mathcal{K}_n(\delta, \varkappa)$ be the least, over mechanisms with dominant-strategy incentives under a retained veto whose expected cost of moves exceeds the least cost by at most $\varkappa$, of the capacity of Proposition 26 that suffices with probability $1 - \delta$, so that the proposition concerns $\varkappa = 0$; whether, for $\delta < \frac{1}{2}$, $\mathcal{K}_n(\delta, \varkappa)$ stays of order $\sqrt{n}$ for every fixed $\varkappa$, or a bounded loss buys bounded capacity, as it does with verifiable sides in the double auction of McAfee (1992) and as Corollary 27 shows it does here under verifiability at no loss at all, is not known for private favored projects.

Theorem 22 compares two moments at which participation becomes binding rather than characterizing the best mechanism for forming a book, which would choose admission, assignment and terms together. Proposition 28 treats information in the simplest common-value form and measures it by expected value. Theorem 29 and Corollary 30 concern a risk-neutral manager with finitely many experiments and allocations, records whose outcomes depend only on each account's position and the state, and investors who value only the law of their own records, and Propositions 44 and 45 price a single deviation under limited liability and risk neutrality. Proposition 46 measures losses for signed combinations of profiles under constant absolute risk aversion and normality, with a constant cost per record held and dense profiles; long-only vehicles, minimum holdings, sparse holdings and the fixed cost of forming a vehicle change the achievable set and the count of records. The equal-treatment results fix a membership and its entitlement structure, rule out direct payments among members, and take the concession and its cost as given.

The threshold-program and search results fix program terms and assume inclusion monotonicity, and the continuous and Bayesian results of Appendix B fix an indispensable cohort, which Internet Appendix III joins only by freezing a coordinated allocation into a binding offer. Section 7 calculates with an assumed expense of standing ready and reports a sensitivity analysis of one function, not a measurement of what fund charges pay for. Welfare is a sum of certainty-equivalent dollars, in which sure transfers net out and real expenses do not, so that the on-path transfers of Theorem 3(i) are not a social cost while the escrow's carry is; supplier rents, taxes and external effects need their own accounting, and pure Nash equilibrium and budget balance do not establish coalition stability (Juarez, 2013; Birmpas, Markakis and Schäfer, 2019). The paper does not treat a fund's other functions, among them liquidity insurance, tax and legal structure, and limited liability, and the counterfactual against which any new arrangement must be judged is a capable one: partnership capital accounts, contributed-property allocations and partner-specific basis records already preserve differentiated state within shared ownership (Internal Revenue Service, 2025a,b), exchange funds combine shared investment with attributable contribution histories (Morgan Stanley Wealth Management, 2023), and subscription-credit agreements differentiate investors' eligibility, advance rates and concentration limits (Goldman Sachs Middle Market Lending Corp. II, 2022). An institution that writes terms across accounts earns its cost only by delivering what such an incumbent cannot provide as cheaply, and its ownership then matters (Hansmann, 1996).

Investments that need several investors to commit are formed by holding some things in common, and they need not be formed by holding everything in common. Under simultaneous commitments, and unless one investor can form the investment alone, they need support, whose cheapest robust form is contingent capacity: among all terms contingent on who commits, the capacity that makes formation the unique rationalizable outcome must exceed, and with an escrow can come arbitrarily close to, the smallest, over coalitions able to form the investment, of their members' expenses of standing ready less the largest of those expenses, and an order substitutes for it among investors whose earlier commitments bind and who trust one another's future rationality. A set of projects financed together needs principal whose permissions cover it, measured by a minimum cut rather than by aggregate capital, and discretion to absorb the mismatch between what investors permit and what projects require, whose amount has the structure of diversification and whose expected cost stays bounded when many investors grant priced permission and their mandates differ idiosyncratically. It also needs decisions surrendered at defined times: investors who keep the right to decide after the terms of a book are known must be backed by replacement capital whose size is set by the tail of a shock common to all of them, while investors who accept the burden before that shock is known, with the transfers it generates returned within the book, leave a shortfall that does not grow with their number. Whoever arranges such investments should therefore bind participation before the common imbalance is known, price the crowded position, return the premium within the book, with each investor's own report left out where exact incentives are required, and hold replacement capital to a quantile of the shortfall, of order $\sqrt{n}$ under a retained veto and a few invitations under binding enrollment. Exposure enters only through who is willing to commit, so that proportional exposure is not required; it economizes on records and on incentive pay, and what it costs investors whose circumstances differ from the population's rises with the dispersion of their optimal holdings. The book is therefore the theory's prediction where mandates differ idiosyncratically, where a pool pays for discretion in proportion to its size and a book pays a bounded amount, and the fund keeps its place where a common component moves mandates together, since both then pay in proportion to size and the economies of records and incentive pay decide.

The forces that technology sets in motion act on these quantities in different directions. Cheaper separate accounts raise the outside values against which investors decide, and where the formation game has completion they compress the range above the floor, shrink what assurance can recover and can remove from equilibrium an investment that remains jointly valuable, while equal-treatment clauses make the distributive repair expensive, and cheaper records make individual exposure affordable. Cheaper collective search widens the pool from which shared investments draw their members, finds complementary permissions and compatible substitutes, and lowers the expense of standing ready, but it raises the cost of discretion when it leads many investors to the same recommendation. Whether cheaper search and organization expand the set of investments that can be executed, or mainly intensify competition for investments that would have formed anyway, is the empirical question on which the institutional implications turn. The investment frontier that investors attain is decided where these forces meet, at the investor whose commitment

is pivotal, in the composition of the population's mandates, and in the moment at which their decisions become binding.

A Proofs

Table 3 collects the notation of Sections 2 to 5; symbols introduced in later sections and in Appendix B are defined where they are used.

Proposition 1

Let Y_i be the set of holdings investor i can attain alone. Because every claim is available to him separately in any quantity on the terms any arrangement would offer, and because no claim requires a formation cost, a minimum scale or scarce capacity, a holding that a joint plan assigns to him belongs to Y_i and costs him the same, whatever others hold. Because his payoff depends only on his own holding, it is the same in either arrangement. The attainable joint allocations are therefore $\prod_i Y_i$. □

Theorem 3

Let investor i receive $t_i \geq 0$ if he commits and $\beta_i \geq 0$ if he commits and the investment fails, write $d_i = t_i + \beta_i$, which covers own payments ($\beta = 0$), escrowed payments ($t = 0$) and their combination, and let C be the set of investors with $d_i > h_i$. Relative to refusing, a committed investor gains $g_i(S) + t_i > 0$ if the committed set S forms the investment and $d_i - h_i$ if it does not. An investor in C therefore gains strictly from committing whatever others do and commits with probability one in every equilibrium, mixed and Bayesian equilibria included. If $C \in \mathcal{F}$ the investment forms. Otherwise, if $C \cup \{j\} \in \mathcal{F}$, investor j's commitment forms it, because $\mathcal{F}$ is closed under enlargement, and yields a strictly positive gain, so j commits and the investment forms. Every other investor then gains by committing, because the investment forms whether or not he does. Conversely, if neither C nor any $C \cup \{j\}$ belongs to $\mathcal{F}$, let exactly the members of C commit. They obtain $d_i - h_i > 0$, the most available to them since the investment fails without further commitments, and an outsider who commits cannot form it and obtains $d_j - h_j \leq 0$, so refusal is a best response. This is an equilibrium in which the investment fails.

For the bounds, let d implement formation and choose an inclusion-minimal $F \in \mathcal{F}$ with at most one member outside C; call that member j, or let j be a most expensive member of F if there is none. Then $F \setminus \{j\} \subseteq C$ and $F \setminus \{j\} \notin \mathcal{F}$, so

$$\sum_{i \in F \setminus \{j\}} d_i \;\geq\; \sum_{i \in F \setminus \{j\}} h_i \;\geq\; \sum_{i \in F} h_i - \max_{i \in F} h_i \;\geq\; R^*,$$

with the first inequality strict if F has at least two members. Under own payments the members of C commit in every equilibrium, so the payments to them are made, and they are at least the left side. Under escrowed payments nothing is paid in equilibrium, and the left side is a claim that the failing coalition $F \setminus \{j\}$ can present, so $R(\beta)$ is at least as large. For attainment, note that if $F' \subset F$ with both in $\mathcal{F}$, then $\sum_{F'} h - \max_{F'} h \leq \sum_F h - \max_F h$, so a minimizer F^* of R^* can be

Table 3: Principal notation of Sections 2–5

Symbol	Meaning
N, n	Investors and their number
o_i; L, H, G	Outside value; worst and best equilibrium welfare and the selection gap, (4)
$\mathcal{F}$	Formation structure: the committed sets that form the investment
h_i	Expense of standing ready, borne by a committed investor and wasted if the investment fails
$g_i(S)$	Net gain of a member when the committed set $S \in \mathcal{F}$ forms
t_i, β_i	Own payment upon committing; escrowed payment upon committing and failure
$b_i(S)$	Payment of a general scheme at a failing committed set S
$R(\beta)$, $R(b)$	Liability: the largest total owed at a failing profile, (8) and (9)
R^*	$\min_{F\in\mathcal{F}}(\sum_F h - \max_F h)$, the least liability under rationalizability
$\nu_{\min}$, R_c	Size of the smallest forming coalition; sum of the $\nu_{\min}-1$ smallest expenses
C	Capacity of a promisor backing several investments
ℓ, N_r	Number of public rounds; investors assigned to round r
R^*_ℓ, $\hat{R}_\ell$	Least liability with ℓ rounds: time-invariant escrow, (10); staged escrow, (11)
p_i, p_{-i}, $p_{>k}$, $p_{\leq k}$	Probability of willingness and its products over the others, those after k and those up to k
f_i, $\underline{g}_i$	Charge (premium or fee); least gain of a willing type
Ω, E, c_e	States of nature; experiments available to a manager; cost of experiment e
π, τ; μ^π_i, μ^π	A plan and the target plan; laws of account i's record and of all records under π
ϕ_i, Ψ; ϱ	Own-account and joint compensation; smallest likelihood ratio of a deviation to the target
V_i, $\underline{\phi}_i$	Investor i's gain from the target; least expected payment that motivates the manager for account i alone
ι, $\bar{\tau}$, c	Annual cost of holding capacity; length of the closing window; cost of delaying formation by a round, whose length is also written τ
q_i, K_j, Γ_i	Investor i's commitment; project j's requirement of capital; the projects i permits
$\mathcal{N}(\mathcal{J}')$, B^*	Investors permitted to finance some project in $\mathcal{J}'$; least compatible capital to be added, Proposition 16
$\mathcal{W}$, $\mathcal{F}(\Gamma,\mathcal{W})$, $R^*(\Gamma,\mathcal{W})$	Willing investors; formation structure and least liability of a book with permissions Γ
ρ_j, ψ, f, $f_n(\delta)$	Requirement shares; shares of narrow mandates; flexible share (distinct from the charge f_i); least flexible share that finances the program with probability $1-\delta$
η	Share of narrow mandates following a common recommendation
c_i, G_c, g_c, φ	Cost of being moved off the favored project, its distribution and density; share of flexible investors
D_j, C_n, ϖ_n	Over-subscription; least total cost of moves; price of the crowded project
v_i, G_v, g_v; γ, Z	Participation margin of an investor, its distribution and density; the ratio $g_v(0)/g_c(0)$; a standard normal variable
S_n, Q_n, Q'_n	Replacement capital; shortfall under a retained veto and under binding enrollment
v^*_n, a^*, μ_v	Enrollment threshold; limit of $Q'_n/\sqrt{n}$ with the premium retained; mean of the limiting Poisson shortfall with the premium returned
ϖ^{-i}_n, Λ_n	Price without investor i's report; premium not returned under the leave-one-out rebate
ϖ°_i, $\bar{\varpi}^\circ_n(c)$, κ_v, λ_v	Price investor i faces; its expectation counted above c; least rate at which a charge loses investors per unit of revenue; $\min\{\kappa_v, g_v(0)/2\}$
τ_0/n, $\underline{v}_n$, T_n, Δ_n, $\mathcal{S}_n(\delta)$	Fee per place; margin below which a claim of zero cost gains under a veto; transfers collected from accepting investors; cash the mechanism must find; capacity that suffices with probability $1-\delta$
n_{eff}	Effective number of commitments, $(\sum_i q_i)^2/\sum_i q_i^2$
ε, $\bar{d}(\varepsilon)$	Informativeness of mandates; band of imbalances that discretion should absorb
k, T, κ, ν_j	Number of exposure profiles; classes of investors who cannot share a vehicle; cost of a record; singular values

chosen inclusion-minimal. Let j^* be a most expensive member and set $d_i = h_i + \varepsilon$ for $i \in F^* \setminus \{j^*\}$ and $d_i = 0$ otherwise. Then $C = F^* \setminus \{j^*\}$ and $C \cup \{j^*\} = F^* \in \mathcal{F}$, so the schedule implements formation; the own payments made in equilibrium to C, and the escrowed liability, which is the claim of the failing coalition C because payments are confined to it, both equal $R^* + (|F^*| - 1)\varepsilon$.

(iii) If $\{a\} \in \mathcal{F}$, then $R^* = 0$ and the null schedule has $C = \emptyset$ with $C \cup \{a\} \in \mathcal{F}$, so it implements formation at no cost and with no liability. If no singleton forms the investment, every F in the bound has at least two members, the first inequality is strict, and both the payments and the liability exceed R^*. □

Theorem 4

Fix a payment scheme that implements formation robustly. With two actions an action is never a best response exactly when the other strictly dominates it, and the set of profiles that survive iterated removal of strictly dominated actions does not depend on the order of removal; fix any order and list the investors whose refusal is removed as $i_1, i_2, \ldots$ in the order of removal, ordering arbitrarily those removed at the same stage. An investor not on the list can refuse in every surviving profile, so the profile in which exactly the listed investors commit survives, and it forms the investment. Let K be the first index for which $A_{K+1} = \{i_1, \ldots, i_K\} \in \mathcal{F}$, so that $A_K = \{i_1, \ldots, i_{K-1}\} \notin \mathcal{F}$; A_K is nonempty because no single investor forms the investment. For $j < K$, when the refusal of i_j was removed, the profile in which the members of A_K other than i_j commit and everyone else refuses had not been removed: investors listed before i_j could only commit, those listed after him could still do either, and investors not in A_K could still refuse. Against that profile, committing gives i_j the payoff $b_{i_j}(A_K) - h_{i_j}$, because A_K fails, and refusing gives $b_{i_j}(A_K \setminus \{i_j\}) \geq 0$, because that set fails as well. Strict dominance therefore requires $b_{i_j}(A_K) > h_{i_j}$, and

$$R(b) \;\geq\; \sum_{i \in N} b_i(A_K) \;\geq\; \sum_{j<K} b_{i_j}(A_K) \;>\; \sum_{i \in A_K} h_i \;=\; \sum_{i \in A_{K+1}} h_i - h_{i_K} \;\geq\; \sum_{i \in A_{K+1}} h_i - \max_{i \in A_{K+1}} h_i \;\geq\; R^*.$$

Payments at formed profiles enter nowhere. For attainment, the escrowed schedule in the proof of Theorem 3, which pays $h_i + \varepsilon$ to the members of an inclusion-minimal minimizer F^* other than a most expensive member j^*, makes committing strictly dominant for those members, then for j^*, whose commitment forms the investment, and then for every other investor, for whom the investment forms whatever he does and membership pays strictly; formation is the only surviving profile, nothing is paid at it, and the liability is $R^* + (|F^*| - 1)\varepsilon$. □

Theorem 5

Write $u_i(S)$ for investor i's payoff when the committed set is S, so that a member of a formed set obtains $g_i(S) > 0$, a committed member of a failing set obtains $b_i(S) - h_i$, and an investor who declines obtains zero. Formation is the unique correlated equilibrium exactly when the linear program that maximizes the probability of failing profiles over correlated equilibria has value

zero. By linear-programming duality, the dual has a weight $\lambda_{i,a} \geq 0$ for each investor and each recommended action, and value zero means that for every profile S,

$$\sum_{i \in S} \lambda_{i,1}(u_i(S \setminus \{i\}) - u_i(S)) + \sum_{i \notin S} \lambda_{i,0}(u_i(S \cup \{i\}) - u_i(S)) \;\geq\; \mathbf{1}\{S \notin \mathcal{F}\}.$$

At $S = N$, which forms, every term in the first sum is $-g_i(N) < 0$, since an investor who declines is paid nothing whether or not $N \setminus \{i\}$ forms, and the second sum is empty, so $\lambda_{i,1} = 0$ for every i. Write $\lambda_i = \lambda_{i,0}$ and $\Lambda = \{i : \lambda_i > 0\}$. At a failing S the inequality reads $\sum_{j \notin S} \lambda_j(u_j(S \cup \{j\}) - 0) \geq 1$, so $\Lambda \in \mathcal{F}$, since otherwise the inequality at $S = \Lambda$ would have a left side of zero. Call a failing set S *deep* if $S \cup \{j\}$ fails for every j; at a deep S the inequality reads

$$\sum_{j \notin S} \lambda_j(b_j(S \cup \{j\}) - h_j) \;\geq\; 1.$$

The same inequality holds, with weights scaled by marginal utilities, when investors are risk averse. Let investor j have a strictly increasing concave utility U_j of wealth, let W_j be his wealth when he declines, which may be random but is independent of all investors' actions and of the correlating device, let the expected utilities below be finite, and let $s_j = \mathbb{E}[U_j'(W_j)]$, with U_j' the right derivative, be finite; it is positive because U_j is strictly increasing and concave. The dual inequalities then hold with expected utilities in place of monetary payoffs; at $S = N$ committing is still strictly preferred, provided that each member's gain from formation is positive in expected utility, as it is when it is a sure gain, so $\lambda_{i,1} = 0$; and at a deep S, since $U_j'(w)$ is a supergradient of U_j at w, so that $U_j(w + x) - U_j(w) \leq U_j'(w)x$ for every x, concavity gives $\mathbb{E}[U_j(W_j + b_j - h_j)] - \mathbb{E}[U_j(W_j)] \leq s_j(b_j - h_j)$, so that the inequality above holds with $\lambda_j s_j$ in place of λ_j. The support of these weights is that of λ, and the argument below uses only the inequality at deep sets and that support, so it applies unchanged. Partition the investors into sets O and M, fix $k \geq 0$, and suppose that $O \cup A$ fails for every $A \subseteq M$ with $|A| \leq k + 1$, so that every such $O \cup A$ with $|A| = k$ is deep and its entrants are the members of $M \setminus A$; write $\pi(B) = \prod_{i \in B} \lambda_i$, with $\pi(\emptyset) = 1$. Multiplying the inequality at $O \cup A$ by $\pi(A)$ and summing over the sets $A \subseteq M$ of size k, each pair of a set A and an entrant $j \in M \setminus A$ corresponds to the set $B = A \cup \{j\}$ of size $k + 1$ and the member $j \in B$, and $\pi(A)\lambda_j = \pi(B)$, so

$$\sum_{B \subseteq M,\, |B| = k+1} \pi(B) \sum_{j \in B} (b_j(O \cup B) - h_j) \;\geq\; \sum_{A \subseteq M,\, |A| = k} \pi(A).$$

If $\Lambda \cap M$ has at least $k + 1$ members, the right side and $\sum_B \pi(B)$ are positive. The total paid at the failing profile $O \cup B$ is at least $\sum_{j \in B} b_j(O \cup B)$ and at most $R(b)$, so

$$R(b) \;>\; \frac{\sum_B \pi(B) \sum_{j \in B} h_j}{\sum_B \pi(B)} \;\geq\; \min_{B \subseteq M,\, |B| = k+1} \sum_{j \in B} h_j.$$

For the general bound take $O = \emptyset$, $M = N$ and $k = \nu_{\min} - 2 \geq 0$: every set of $\nu_{\min} - 1$ investors fails, $\Lambda \in \mathcal{F}$ has at least $\nu_{\min} = k + 2$ members, and the minimum is R_c. For a count $\mathcal{F} = \{S : |S \cap M| \geq \nu\}$ take $O = N \setminus M$ and $k = \nu - 2$, so that $O \cup A$ fails whenever $|A| \leq \nu - 1$. If $\Lambda \cap M$ had at most $\nu - 1$ members, $O \cup (\Lambda \cap M)$ would fail and the inequality there would have a left side of zero, so $\Lambda \cap M$ has at least ν members. The minimum is the sum of the $\nu - 1$ smallest expenses in M. A coalition in $\mathcal{F}$ contains at least ν members of M, and adding investors to a coalition never lowers its expenses less the largest, so R^* is attained by the ν members of M with the smallest expenses and equals that sum. For a closed cohort the sets B are $F_0 \setminus \{m\}$ for $m \in F_0$, and $\pi(B)$ is proportional to $1/\lambda_m$.

For the example in the text, the weights $\lambda = (1, 1, 6, 6)$ give, at the failing profiles $\emptyset$, $\{1\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{3, 4\}$ and $\{1, 3, 4\}$, the values 7, $\frac{9}{20}$, $\frac{3}{10}$, 240, 2, $\frac{1}{5}$ and 20, and the others follow by symmetry, so formation is the unique correlated equilibrium. With gains of 14, the linear program for the largest probability of failure has value one. No action is strictly dominated: investors 1 and 2 lose 0.75 at a pair of their kind and gain 0.5 alone, and investors 3 and 4 lose their expense when both of them and one required investor have committed and gain 0.5 alone. For the count of three seats among four, the weights $\lambda_{A,0} = 5$, $\lambda_{B,0} = \lambda_{C,0} = \frac{1}{25}$, $\lambda_{D,0} = 1$ and $\lambda_{D,1} = 11$ certify uniqueness, the positive weight on D's recommendation to commit being possible because D is indifferent at N; the scheme pays 1.1 to A, B or C alone and to D in each pair he forms with another, $\frac{1}{2}$ to D alone, 0.99 and 0.1 to A and B in the pair $\{A, B\}$, 0.995 and 0.1 to A and C in $\{A, C\}$, 1.1 to C in $\{B, C\}$, and $\frac{1}{100}$ to D when he declines and the others form, with gains of 20 to A, B and C and $\frac{1}{100}$ to D at formed profiles. □

Corollary 6

The sufficiency argument uses only the signs of members' net gains at formed profiles, which fees smaller than the gains leave positive, and the liability depends only on payments at failing profiles. Fees approaching each member's gain at the formed profile transfer nearly the whole surplus of formation to the promisor, who pays nothing at that profile. Theorem 4 bounds the required capacity by R^* from below whatever is paid at formed profiles, and R^* involves only expenses of standing ready. □

Proposition 7

If formation is not the unique rationalizable outcome, some surviving profile fails, and since $N \in \mathcal{F}$ some investor refuses in it, so that his refusal survives. In the game with payoffs known to all, the surviving actions of an investor are the interim correlated rationalizable actions of the type that holds common knowledge of the payoffs (Dekel, Fudenberg and Morris, 2007). Under the richness assumption, Proposition 1 of Weinstein and Yildiz (2007) states that for every type and every interim correlated rationalizable action of that type there are types converging to it in

the product topology, each with a finite, dominance-solvable model, for which that action is the unique rationalizable action; applied to the refusal identified above, it gives the statement.

In the example, write a, b and c for the probabilities with which the three investors commit. The gains from committing over refusing are $V_1 = 3bc+(1-b)(1-c)-b(1-c)-(1-b)c = 1-2b-2c+6bc$, $V_2 = 3ac + \frac{1}{4}a(1-c) - 2(1-a)$ and $V_3 = 5ab - 2$. Universal commitment is an equilibrium, and no other pure profile is: with nobody else committing the first investor gains 1, with only the first committed the second gains $\frac{1}{4}$, with the first two committed the third gains 3, and in every other failing profile some committed investor loses his expense. A fully mixed equilibrium requires $ab = \frac{2}{5}$ and $a = 2/(\frac{9}{4} + \frac{11}{4}c)$, hence $b = \frac{9}{20} + \frac{11}{20}c$, and then $V_1 = 0$ becomes $3.3c^2 - 0.4c + 0.1 = 0$, which has no real root. If the third investor refuses for certain, indifference of the other two requires $b = \frac{1}{2}$ and $a = \frac{8}{9}$, so that $ab = \frac{4}{9} > \frac{2}{5}$ and the third strictly prefers to commit; if he commits for certain, it requires $b = \frac{1}{4}$ and $a = \frac{2}{5}$, so that $ab < \frac{2}{5}$ and he strictly prefers to refuse. If the first investor commits for certain, the second strictly prefers to commit, since $V_2 = \frac{1}{4} + \frac{11}{4}c > 0$, and then the third does; if he refuses for certain, the second and third strictly prefer to refuse and the first then strictly prefers to commit. If the second investor commits for certain and the others mix, $a = \frac{2}{5}$ and $c = \frac{1}{4}$, at which $V_2 < 0$; if he refuses for certain, the third refuses and the first commits, at which $V_2 > 0$. The remaining profiles, in which exactly one investor mixes, require an indifference that these comparisons exclude. Formation is therefore the only Nash equilibrium. The linear program of the proof of Theorem 5 gives a correlated equilibrium that places probability $813/815 \approx 0.9975$ on failing profiles, the most that any correlated equilibrium places there. No action is strictly dominated, since each investor gains from committing when the others' commitments form the investment and the first loses when exactly one other has committed, the second when only the third has, and the third when the investment fails, while the first gains from committing when nobody else does; every profile is therefore rationalizable. □

Proposition 8

Order the investments by seniority. For sufficiency, let $C > \max_k R_k^*$ and give each investment the escrowed schedule of Theorem 3 with margins small enough that each schedule's liability is below C. A claim of the most senior investment is paid in full whatever happens elsewhere, so the iterated dominance of Theorem 4 applies to its investors in the joint game and its formation is the only surviving outcome; at every surviving profile it then makes no claim, so the claims of the second investment are paid in full at every surviving profile, and the same argument applies to it; and so on down the order. Formation of every investment is the only surviving outcome, and nothing is paid. For necessity, apply the argument of Theorem 4 to the investors of investment k in the joint game, and fix a profile σ_{-k} of the other investments' investors that survives the whole procedure, at which, by hypothesis, every other investment forms and makes no claim. Because the sets of surviving actions only shrink, σ_{-k} was available at every stage. When the refusal of the investor i_j is removed, committing must beat refusing against the profile in which the members of A_K in investment k other than i_j commit, the others in investment k refuse, and the

other investments' investors play σ_{-k}, so the payments actually delivered to the members of A_K at the single joint profile in which they commit and the others play σ_{-k} exceed their expenses. Those deliveries are paid out of C, so $C > \sum_{i \in A_K} h_i \geq R_k^*$. Separate promisors face the requirement of Theorem 4 investment by investment. □

Proposition 9

Let investors move in a fixed order, each observing the commitments made before him, and call a history *viable* if the investors who have committed, together with all who have yet to move, form a coalition in $\mathcal{F}$. The claim is that every subgame-perfect continuation from a viable history forms the investment, and that every outcome surviving backward iterated dominance from a viable history does so; both are proved by backward induction on the number of moves already made, since the comparisons below hold whatever the surviving behavior after the mover's choice. When all n have moved, a viable history is one whose committed set belongs to $\mathcal{F}$. Suppose the claim holds at every viable history after $k+1$ moves, and consider the next mover at a viable history after k moves. If the history remains viable when he declines, the investment forms whatever he does, by the induction hypothesis, and he commits because membership pays him strictly. Otherwise only his commitment keeps the history viable, which by the induction hypothesis forms the investment and pays him strictly, while declining leaves a history from which the investment cannot form and pays him nothing; he commits. The empty history is viable because $N \in \mathcal{F}$, and no payment is made at any history. □

Proposition 10

Let $N_1, \ldots, N_\ell$ be the sets of investors assigned to the rounds, C the assured investors and $U = N \setminus C$ the others. Relative to refusing, an assured investor gains $g_i(S) + t_i > 0$ if the final committed set S forms the investment and $t_i + \beta_i - h_i > 0$ if it does not, so in every subgame-perfect equilibrium he commits at every history, while an unassured investor who commits gains $t_i + \beta_i - h_i \leq 0$ if the investment fails. Call a set $\sigma \subseteq U$ a *selection* if it contains at most one investor from each round.

Sufficiency. Let $C \cup \sigma \in \mathcal{F}$ for a selection σ, let u_k be its member in round k where there is one, and call a history at the start of round k *favorable* if every assured investor of an earlier round and every u_j with $j < k$ has committed. Every subgame-perfect continuation from a favorable history forms the investment, by backward induction on k. After the last round a favorable history has a committed set containing $C \cup \sigma$, which forms the investment. Suppose the claim holds at favorable histories at the start of round $k+1$ and consider one at the start of round k. The assured investors of round k commit, because they do so at every history. If round k contains no member of σ, the next history is therefore favorable whatever the others do. Otherwise, if u_k commits, the next history is favorable whatever the unassured others in his round do, the investment forms and he gains strictly, while refusing yields his outside value, so he commits. The empty history is favorable. The same comparisons hold whatever the remaining behavior at later histories, so the

commitments of assured investors and of each u_k at favorable histories survive backward iterated dominance and their refusals do not, and the schedule implements formation by backward iterated dominance as well.

Necessity. Suppose no selection σ has $C \cup \sigma \in \mathcal{F}$, and call a history a *selection history* if the unassured investors who have committed form a selection. Consider the strategies in which assured investors commit at every history, unassured investors refuse at every selection history, and play at every other history follows a subgame-perfect equilibrium of the subgame it begins, which exists in behavior strategies because the subgame is finite; these continuations are well defined, because every successor of a history that is not a selection history is also not one. At a selection history in round k, if a single unassured investor i of that round deviates and commits, the committed unassured investors still form a selection, every later history on the continuation path is a selection history, and the final committed set is $C \cup \sigma'$ for a selection σ', which fails; i obtains $t_i + \beta_i - h_i \leq 0$, no more than by refusing. Assured investors' commitments are strictly optimal at every history. By the one-shot deviation principle these strategies form a subgame-perfect equilibrium, and on its path the final committed set is C, which fails.

Bounds. Let a schedule implement formation, choose a selection σ with $C \cup \sigma \in \mathcal{F}$, and choose an inclusion-minimal $F \in \mathcal{F}$ contained in $C \cup \sigma$. At most ℓ members of F lie outside C, one from each round. If some do, $F \cap C$ is a proper subset of F and fails, and

$$\sum_{i \in F \cap C} (t_i + \beta_i) \;\geq\; \sum_{i \in F \cap C} h_i \;=\; \sum_{i \in F} h_i - \sum_{i \in F \setminus C} h_i \;\geq\; \sum_{i \in F} h_i - h_F^{(\ell)} \;\geq\; R_\ell^*,$$

with the first inequality strict if $F \cap C$ is nonempty; if none does, removing a most expensive member of F leaves a failing subset of C, and the same chain holds because $\max_F h \leq h_F^{(\ell)}$. Under an escrowed schedule the left side is a claim that a failing coalition can present, so the liability is at least R_ℓ^*; under an own schedule the assured investors commit in every equilibrium, and the left side is paid. For attainment, let F^* attain R_ℓ^*, let U^* consist of the $\min\{\ell, |F^*|\}$ most expensive members of F^*, place one of them in each of the first $|U^*|$ rounds and every other investor in the first round, and pay $h_i + \varepsilon$ to each $i \in F^* \setminus U^*$ and nothing to anyone else. Then $C = F^* \setminus U^*$, and U^* is a selection with $C \cup U^* = F^* \in \mathcal{F}$, so the schedule implements formation; its liability, which is the claim of the failing coalition C because payments are confined to it, and the own payments it makes in equilibrium both equal $R_\ell^* + |C|\varepsilon$. If F^* has at most ℓ members, C is empty and the schedule pays and promises nothing. Conversely, a schedule with zero liability or zero payments in equilibrium has $C = \emptyset$, since a nonempty C makes the chain above strict, so some selection, of at most ℓ investors, forms the investment. With one round a selection has at most one member and $h_F^{(1)} = \max_F h$, so $R_1^* = R^*$, and $h_F^{(\ell)}$ rises with ℓ, so R_ℓ^* falls weakly. For a fixed assignment the same chain holds with $\sum_{i \in F \setminus C} h_i \leq \sum_r \max_{i \in F \cap N_r} h_i$, and leaving unassured the most expensive member of each round within a minimizing coalition attains $\min_{F \in \mathcal{F}} \left(\sum_{i \in F} h_i - \sum_r \max_{i \in F \cap N_r} h_i \right)$. □

Theorem 11

With the assignment fixed and commitments observed, a history is determined by the committed set, so a scheme contingent on the profile of commitments is contingent on the history. Write $F_r = F \cap N_r$, and for a history H let $O(H)$ be the set of outcomes that survive backward iterated dominance from H; at a history at which the remaining actions of the movers are given, O of the history is the union of O of its successors under those actions.

Sufficiency. Let F attain $\hat{R}_\ell$, let p_r be a most expensive member of F_r when F_r is nonempty, and let the staged escrow pay $h_i + \varepsilon$ to a committed $i \in F_r \setminus \{p_r\}$ if r is the first round in which the members of F did not all commit and the investment fails. Call a history at the start of round k *complete* if every member of F assigned to an earlier round has committed. Every outcome in $O(H)$ forms the investment when H is complete, by backward induction on k. After the last round a complete history contains F. At a complete history at the start of round k, a member $i \in F_k \setminus \{p_k\}$ gains strictly from committing whatever the others do and whatever survives later: if all of F_k commit, the next history is complete and every surviving outcome forms, which pays him a positive gain; otherwise round k is the first incomplete round, and he receives ε net if the investment fails and a positive gain if it forms, while declining yields his outside value. His refusal is removed. Given that, p_k's commitment completes the round, so that every surviving outcome forms, whereas declining yields his outside value; his refusal is removed, and the next history is complete whatever the others in the round do. If F_k is empty the next history is complete whatever is done. The empty history is complete. The same comparisons show that the schedule implements formation in every subgame-perfect equilibrium. At a failing profile only the members of the first incomplete round are paid, so the liability is at most $\max_r \sum_{i \in F_r \setminus \{p_r\}} (h_i + \varepsilon) \le \hat{R}_\ell + n\varepsilon$.

Lower bound. Let a scheme implement formation by backward iterated dominance, so that every outcome in $O(\emptyset)$ forms. Suppose every outcome in $O(H_k)$ forms for a history H_k at the start of round k. List the members of N_k whose refusal is removed at H_k as $j_1, j_2, \dots$ in the order of removal. Investors of round k whose refusal is not removed can still decline, so the history $H_k \cup \{j_1, \dots, j_d\}$, at which exactly the listed members commit, is reached by remaining actions, and every outcome in its O forms. Let m^* be the least m such that every outcome in $O(H_k \cup \{j_1, \dots, j_m\})$ forms. If $m^* = 0$, let P_k be empty and $H_{k+1} = H_k$. Otherwise write $P_k = \{j_1, \dots, j_{m^*-1}\}$ and $q_k = j_{m^*}$, and choose a failing outcome $T_k \in O(H_k \cup P_k)$, which exists by the choice of m^*. For $j_i \in P_k$, when his refusal at H_k was removed, the round-k actions in which the other members of P_k commit and every other investor of the round declines were still available, by the argument in the proof of Theorem 4, and so was the later behavior leading to T_k, because the sets of remaining actions only shrink. Against those actions, committing leads to T_k and gives $b_{j_i}(T_k) - h_{j_i}$, while declining gives a nonnegative payoff. Hence $b_{j_i}(T_k) > h_{j_i}$ for every member of P_k, and $R(b) > \sum_{i \in P_k} h_i$ when P_k is nonempty. Let $H_{k+1} = H_k \cup P_k \cup \{q_k\}$, every outcome in whose O forms, and repeat; backward iterated dominance operates at every history, reached or not, so the

argument applies at H_{k+1}. After the last round, $F = H_{\ell+1}$ forms the investment, its members in round k are $P_k \cup \{q_k\}$, and

$$R(b) \;\geq\; \max_k \sum_{i \in P_k} h_i \;\geq\; \max_k \Big(\sum_{i \in F_k} h_i - \max_{i \in F_k} h_i \Big) \;\geq\; \hat{R}_\ell,$$

with strict inequality whenever some P_k is nonempty, which is the case unless every F_k has at most one member, and then $\hat{R}_\ell = 0$. □

Proposition 12

With every investor indispensable, N is the only forming coalition, and the staged escrow for N pays $h + \varepsilon$ to each committed member of round r other than one designated member if round r is the first in which not every member committed. A round in which some member refuses makes formation impossible, so a claim of round r is determined when that round closes, and if the round completes no claim of that round can arise later. The escrow of round r need therefore be available only while round r is open, in the amount $(|N_r| - 1)(h + \varepsilon)$, and its carry is $\iota\tau(|N_r| - 1)(h + \varepsilon)$; summing over the rounds gives $\iota\tau(n - \ell)(h + \varepsilon)$ whatever their sizes, and Theorem 11 gives implementation. With the cost of delay the total is $\iota\tau(n - \ell)h + (\ell - 1)c$ plus $\iota\tau(n - \ell)\varepsilon$, which is affine in ℓ with slope $c - \iota\tau h$. Over $\ell \in \{1, \ldots, n\}$ it is least at $\ell = 1$ when the slope is positive and at $\ell = n$, where no escrow is needed, when it is negative. □

Theorem 13

Types are independent, so each type's belief about the others' types is the prior. For a type t_i of investor i, write $\Delta_i(t_i, a_{-i})$ for his payoff from committing less his payoff from refusing when the others' actions are a_{-i}. With two actions, interim correlated rationalizability (Dekel, Fudenberg and Morris, 2007) removes the refusal of t_i at a stage exactly when $\sum_{t_{-i}} \mathbb{P}(t_{-i}) \min_{a_{-i}} \Delta_i(t_i, a_{-i}) > 0$, and his commitment exactly when $\sum_{t_{-i}} \mathbb{P}(t_{-i}) \max_{a_{-i}} \Delta_i(t_i, a_{-i}) < 0$, the minimum and maximum being taken over the others' actions that survive the earlier stages at their types.

(i) Suppose willing formation were the unique rationalizable outcome. Since every other investor is willing with positive probability, an unwilling type whose commitment survived could complete the investment, and a willing type whose refusal survived could prevent it; so in the end willing types only commit and unwilling types only refuse, and some refusal is removed. Consider the first stage at which a refusal is removed, of a type t_i of investor i. At that stage every type of every other investor can still refuse, and willing types can still commit. For a profile of the others' types, let W be the set of those who are willing. Committing at a failing profile pays i the amount $b_i(T \cup \{i\}) - h_i$, whatever his type, so for every $T \subseteq W$ with $T \cup \{i\} \notin \mathcal{F}$, $\min_{a_{-i}} \Delta_i \leq m(W)$, where $m(W) = \min_{T \subseteq W,\, T \neq N \setminus \{i\}} \big(b_i(T \cup \{i\}) - h_i\big)$. Removal therefore requires $\sum_W \mathbb{P}(W)\, m(W) > 0$. Let $A = \sum_{W \neq N \setminus \{i\}} \mathbb{P}(W)(b_i(W \cup \{i\}) - h_i)$. If $m(N \setminus \{i\}) \leq 0$, then $A \geq \sum_{W \neq N \setminus \{i\}} \mathbb{P}(W) m(W) > 0$; if $m(N \setminus \{i\}) > 0$, every proper subset T of $N \setminus \{i\}$ has $b_i(T \cup \{i\}) > h_i$, and $A > 0$ because

each $W \neq N \setminus \{i\}$ has positive probability. Now consider an unwilling type of i whose loss λ is below $A/\mathbb{P}(N \setminus \{i\})$. Against the conjecture that the others' willing types commit and their unwilling types refuse, which is supported by surviving actions at every stage, committing yields $A + \mathbb{P}(N \setminus \{i\})(\varsigma_i - \lambda)$, where $\varsigma_i \geq 0$ is the transfer to i at formation, nonnegative because no charge is imposed; this is positive. That type's commitment therefore survives, contradicting the first step. No gap guarantees that such a type has positive probability.

(ii) At the first stage, a willing type of $i \neq j$ gains $\beta_i - h_i > 0$ from committing at every failing profile and at least $\underline{g}_i - f_i > 0$ at the formed one, so his refusal is removed; an unwilling type of j loses $h_j - \beta_j > 0$ at every failing profile and his loss at the formed one, so his commitment is removed. At the second stage, a willing type of j faces commitment by every willing type of the others, so his commitment forms the investment with probability at least p_{-j}, and the least expected gain over the remaining conjectures is at least $p_{-j}\underline{g}_j - (1 - p_{-j})(h_j - \beta_j) > 0$; his refusal is removed. At the third stage, an unwilling type of $i \neq j$ with loss $\lambda \geq 0$ faces formation exactly when all others are willing, since all others' willing types now commit, and otherwise, at best for him, failure; the largest expected gain over the remaining conjectures is $-p_{-i}(\lambda + f_i) + (1 - p_{-i})(\beta_i - h_i) < 0$, so his commitment is removed. Every type now has a single surviving action, and the outcome is willing formation, whatever the distribution of losses. With a fee f_i upon committing in place of the premium, the willing type gains $\beta_i - h_i - f_i > 0$ at every failing profile and at least $\underline{g}_i - f_i > 0$ at the formed one, and the unwilling type's largest expected gain is $-p_{-i}\lambda + (1 - p_{-i})(\beta_i - h_i) - f_i < 0$, so the same steps go through.

(iii) Commitments are observed and willingness is independent across investors, so at every history each investor's belief about the willingness of those after him is the prior. Write $y_k < 0$ for the formation payoff of an unwilling investor k, and call a history at which investors $1, \ldots, k-1$ have all committed *open* for investor k. After any refusal the investment cannot form, and committing yields $\beta_k - h_k \leq 0$. Suppose the condition holds. Investor n, at an open history, gains a positive amount from committing if he is willing and $y_n < 0$ if he is not, so he commits exactly when willing. Suppose that, from every history open for investor $k+1$, every later investor commits exactly when willing. At a history open for investor k, his commitment forms the investment with probability $p_{>k}$, so committing yields $p_{>k}g - (1 - p_{>k})(h_k - \beta_k) > 0$ for a willing type with gain g and $p_{>k}y_k - (1 - p_{>k})(h_k - \beta_k) < 0$ for an unwilling type, while refusing yields zero; he commits exactly when willing, and by induction formation is willing. With $\beta_k = h_k$ the willing type's payoff is $p_{>k}g > 0$ for every positive gain. Conversely, when every willing type of investor k gains g_k and the condition fails, let k be the largest index at which it fails. A willing investor k obtains at most zero from committing at an open history, so refusing at every type is optimal for him; with every investor before him refusing as well, which is optimal because a commitment then leads to failure, and with refusal after any refusal, these strategies form a perfect Bayesian equilibrium in which the investment never forms. Setting $\beta = 0$ gives the statement for an order without payments. If $\beta_k < h_k$, an unwilling investor refuses at an open history whatever his loss, and every investor refuses after a refusal, so a payment is claimed only by a willing investor k who committed at an open history, with probability $p_{\leq k}$, and only when an investor after him

is unwilling, with probability $1 - p_{>k}$; the expected cost is $\sum_{k<n} p_{\leq k}(1 - p_{>k})\beta_k$, and β_k slightly above $\max\{0, h_k - p_{>k}g_k/(1 - p_{>k})\}$, which lies below h_k, meets the condition, so the infimum is as stated. □

Proposition 34

In an order with willing formation in which nobody commits after a refusal, which holds in every equilibrium when $\beta_k < h_k$ for every k because a commitment after a refusal then loses $h_k - \beta_k$, investor k commits with probability $p_{\leq k}$ and his commitment fails with probability $1 - p_{>k}$, so the expected wasted expense is the stated sum. For the order of commitments, let investors a and b occupy adjacent positions, let $\bar{P}$ be the probability that every investor before them is willing and $\bar{Q}$ the probability that every investor after them is. Their wasted expenses are $\bar{P}[p_a(1 - p_b\bar{Q})h_a + p_a p_b(1 - \bar{Q})h_b]$ with a first, and exchanging them changes no other investor's term, since each depends on a and b only through $p_a p_b$. The difference between the orders with a first and with b first is $\bar{P}[p_a h_a(1 - p_b) - p_b h_b(1 - p_a)]$, which is negative exactly when $p_a h_a/(1 - p_a) < p_b h_b/(1 - p_b)$. The formula covers the last position, whose own term vanishes, so an order that is not sorted by increasing $p_i h_i/(1 - p_i)$ can be improved by an exchange of adjacent investors, and the sorted orders, which differ only among ties, minimize the wasted expense. With simultaneous commitments and willing formation, investor i commits with probability p_i and fails with probability $1 - p_{-i}$. In any order, the investor in position k wastes $p_k\, p_{<k}(1 - p_{>k})h_k$, where $p_{<k} = \prod_{j<k} p_j$, and $p_{<k}(1 - p_{>k}) \leq 1 - p_{<k}p_{>k} = 1 - p_{-k}$ because the difference is $1 - p_{<k} \geq 0$; the inequality is strict for $k \geq 2$, and the last investor wastes nothing in the order and $p_n(1 - p_{-n})h_n > 0$ simultaneously. Finally, let round r have at least two members, and suppose no escrowed payment exceeds the corresponding expense. At a history at which some investor has refused, and at a history in round r or an earlier round at which nobody has refused, every investor refuses at every type. At histories in later rounds at which nobody has refused, which are reached only if every member of round r has committed, play is defined backward from the last round: in a round with two or more members every member refuses at every type, and the member of a round with one member commits exactly when his expected payoff from committing, computed with prior beliefs about those who move after him and the play already defined for them, is positive. Beliefs about investors who have yet to move are the prior throughout, which Bayes's rule requires because types are independent and none of their actions has been observed, and beliefs about investors who have moved affect nobody's payoff. Every investor is indispensable, so after a refusal a commitment cannot form the investment and yields at most $\beta_i - h_i \leq 0$, a charge only lowering it, and refusing is optimal. At a history in a round with two or more members at which nobody has refused, another member of the round refuses, so a commitment again cannot form the investment, and refusing is optimal; at such a history in a round with one member before round r, a commitment meets refusal in round r; and the member of a single-member round after r responds optimally by construction. These strategies and beliefs therefore form a perfect Bayesian equilibrium, on whose path the investment never forms. □

Corollary 14

At the first stage no investor is bound, so declining a proposed commitment can be neither charged nor penalized, and by hypothesis the first stage is a formation game with known willingness in which payoffs depend only on the committed set. If its commitments are simultaneous, Theorem 4 applies to it: a scheme that implements formation robustly has liability greater than R^* unless a single investor forms the first-stage commitment alone. If they are made in ℓ public rounds and the scheme implements formation by backward iterated dominance, the lower bound of Theorem 11 applies. □

Proposition 15

Deal by deal, an order with escrowed payments below but close enough to expenses yields willing formation in every perfect Bayesian equilibrium (Theorem 13(iii)), so an investment forms with probability p^n and then yields ng; the escrowed payments are transfers, and the expenses wasted on failed attempts are W_n by Proposition 34, which applies because with $\beta_k < h_k$ nobody commits after a refusal. Expected surplus per investment is $np^n g - W_n$. The pool makes every investment, and each investor's expected value from one is $pg - (1-p)\lambda$, so its expected surplus is $Kn[pg-(1-p)\lambda]-F$. The difference is $K\,D(n)-F$. With one investor $W_1 = 0$ and $D(1) = -(1-p)\lambda$. Let $b(n) = pg(1-p^{n-1}) - (1-p)\lambda$, which increases in n, since $b(n+1) - b(n) = gp^n(1-p) > 0$, and tends to $pg - (1-p)\lambda$, so $\hat{n}$ exists when that limit is positive. W_n increases in n, since

$$\frac{W_{n+1} - W_n}{h} = p^n(1-p) + \sum_{k=1}^{n-1} p^k(p^{n-k} - p^{n+1-k}) > 0,$$

and $W_n > 0$ for $n \geq 2$. For $n \geq \hat{n}$, $b(n) \geq 0$, so $D(n) \geq W_n > 0$, and $D(n+1) - D(n) = b(n+1) + n[b(n+1) - b(n)] + W_{n+1} - W_n > 0$. □

Proposition 35

Fix any information structure and let every investor refuse at every type. A deviating subscriber cannot form the program, because no investor can form it alone, and receives a nonpositive net payment in the failure state. Refusal is therefore a best response at every information set. □

Proposition 16

Form a network with a source, a node for each investor, a node for each project and a sink, with an arc of capacity q_i from the source to investor i, an arc of unlimited capacity from i to each project in Γ_i, and an arc of capacity K_j from project j to the sink. Allocations that respect permissions and commitments are the feasible flows, x_{ij} being the flow from i to j, and the program is financed exactly when the largest flow equals $K(\mathcal{J})$. A cut of finite capacity that leaves the projects $\mathcal{J}'$ on

the sink's side must leave there every investor permitted to finance one of them, and the least such cut has capacity $\sum_{i\in\mathcal{N}(\mathcal{J}')} q_i + K(\mathcal{J}\setminus\mathcal{J}')$. By the max-flow min-cut theorem the largest flow is

$$\min_{\mathcal{J}'\subseteq\mathcal{J}}\Big[K(\mathcal{J}) - K(\mathcal{J}') + \sum_{i\in\mathcal{N}(\mathcal{J}')} q_i\Big] \;=\; K(\mathcal{J}) - \max_{\mathcal{J}'\subseteq\mathcal{J}}\Big[K(\mathcal{J}') - \sum_{i\in\mathcal{N}(\mathcal{J}')} q_i\Big]_+,$$

the positive part arising from $\mathcal{J}' = \emptyset$, which gives the first statement. Capital B permitted to finance every project adds B to $\sum_{i\in\mathcal{N}(\mathcal{J}')} q_i$ for every nonempty $\mathcal{J}'$, so the program becomes financeable exactly when $B \geq B^*$. Widening a permission or adding an investor weakly enlarges every $\mathcal{N}(\mathcal{J}')$, and so weakly raises every sum $\sum_{i\in\mathcal{N}(\mathcal{J}')} q_i$ and weakly lowers B^*. □

Proposition 17

An unwilling investor is not offered the book. Among the willing, if the investors in S can finance the program, so can those in any T with $S \subseteq T \subseteq \mathcal{W}$, by allocating nothing to the members of $T \setminus S$, so $\mathcal{F}(\Gamma, \mathcal{W})$ is closed under enlargement within $\mathcal{W}$; it is nonempty by hypothesis and excludes the empty set because requirements are positive. With the allocation at each closing set chosen by a fixed rule, a committed member's payoff depends only on the committed set, through his gain if it closes the book, which is positive because he is willing, and his expense if it does not, so the commitment game among the willing is a formation game with structure $\mathcal{F}(\Gamma, \mathcal{W})$, and Theorems 3 and 4 give the infimum and its approach by escrowed schedules. (i) $R^*(\Gamma, \mathcal{W})$ is a function of $\mathcal{F}(\Gamma, \mathcal{W})$ and of the expenses, and the allocation rule enters $\mathcal{F}(\Gamma, \mathcal{W})$ only through $\mathcal{W}$, since whether a set can finance the program does not depend on the rule. If $\mathcal{W} \subseteq \mathcal{W}'$, then $\mathcal{F}(\Gamma, \mathcal{W}) \subseteq \mathcal{F}(\Gamma, \mathcal{W}')$, and a minimum over a larger family is no larger. (ii) An allocation that respects Γ respects any wider permissions, so $\mathcal{F}(\Gamma, \mathcal{W}) \subseteq \mathcal{F}(\Gamma', \mathcal{W})$ when $\Gamma_i \subseteq \Gamma'_i$ for every i. (iii) A financing allocation places $K(\mathcal{J}) = \sum_{i\in\mathcal{W}} q_i$ units, while a set $S \subsetneq \mathcal{W}$ commits $\sum_{i\in S} q_i < \sum_{i\in\mathcal{W}} q_i$ because commitments are positive, so $\mathcal{F}(\Gamma, \mathcal{W}) = \{\mathcal{W}\}$ whenever $\mathcal{W}$ can finance the program, and $R^*(\Gamma, \mathcal{W}) = \sum_{i\in\mathcal{W}} h_i - \max_{i\in\mathcal{W}} h_i$. □

Theorem 18

Take $\rho_j n$ and fn to be integers, and let $\bar{n} = (1-f)n$ be the number of narrow investors. Because $\sum_j K_j = n$, the flexible units number $fn = \sum_j (K_j - n_j)$, so if $n_j \leq K_j$ for every j the program is financed by placing every narrow unit in its project and the flexible units in the remaining places; if $n_j > K_j$ for some j, at least $n_j - K_j$ narrow units cannot be used, and since all capital is needed the program cannot be financed. Making one narrow investor flexible lowers one n_j by one, so, coupling the draws, the probability of financing is nondecreasing in f, and $f_n(\delta)$ is well defined.

(i) With two equal projects, n_A is binomial with $\bar{n}$ trials and probability $\frac{1}{2}$, and $n_B = \bar{n} - n_A$, so the program is financed exactly when $|2n_A - \bar{n}| \leq fn$. If $fn = a\sqrt{n}$ for a constant a, then $\bar{n}/n \to 1$, and by the central limit theorem the probability of financing converges to $\mathbb{P}(|Z| \leq a)$ for a standard normal Z; it is eventually below $1-\delta$ when $a < z_{1-\delta/2}$ and above it when $a > z_{1-\delta/2}$,

and by monotonicity $\sqrt{n}\, f_n(\delta) \to z_{1-\delta/2}$. With m projects, n_j is binomial with $\bar{n} \le n$ trials and probability ρ_j, and because $K_j - \bar{n}\rho_j = \rho_j f n$ the program fails only if $n_j - \bar{n}\rho_j > \rho_j f n$ for some j. Bernstein's inequality, with variance at most $n\rho_j$ and centered increments bounded by one, bounds the probability of each such event by

$$\exp\Big(-\frac{(\rho_j f n)^2}{2(n\rho_j + \rho_j f n/3)}\Big) \;\le\; \exp\Big(-\frac{3\rho_j f^2 n}{8}\Big),$$

using $f \le 1$, and the union bound makes the probability of failure at most $m\exp(-3\rho_{\min} f^2 n/8)$, which is at most δ when $(fn)^2 \ge 8n\log(m/\delta)/(3\rho_{\min})$.

(ii) Because $\sum_j \psi_j = \sum_j \rho_j$, some $\psi_j \ge \rho_j$, so $Y \le 1$. Conditionally on ψ, the strong law gives $n_j/n \to (1-f)\psi_j$ almost surely. If $1-f < Y$, every $(1-f)\psi_j$ is below ρ_j and the program is financed for all large n; if $1-f > Y$, some $(1-f)\psi_j$ exceeds ρ_j and it fails for all large n. By dominated convergence, the probability $P_n(f)$ of financing with share f satisfies $\liminf_n P_n(f) \ge \mathbb{P}(Y > 1-f)$ and $\limsup_n P_n(f) \le \mathbb{P}(Y \ge 1-f)$. Let F_Y be the distribution function of Y; continuity at Q_δ gives $F_Y(Q_\delta) = \delta$, and strict increase there gives $F_Y(y) < \delta < F_Y(y')$ for $y < Q_\delta < y'$. If $1-f'' < Q_\delta$, then $\liminf_n P_n(f'') \ge 1 - F_Y(1-f'') > 1-\delta$, so $f_n(\delta) \le f''$ for all large n. If $1-f' > Q_\delta$, choose y' strictly between Q_δ and $1-f'$; then $\limsup_n P_n(f') \le 1 - F_Y(y') < 1-\delta$, so $f_n(\delta) > f'$ for all large n. Hence $f_n(\delta) \to 1 - Q_\delta$. With two equal projects $Y = 1/(2\max\{\psi_1, 1-\psi_1\}) = 1/(1+|2\psi_1 - 1|)$, a decreasing function of $|2\psi_1 - 1|$, so $Q_\delta = 1/(1+\zeta_\delta)$ and $f^*(\delta) = \zeta_\delta/(1+\zeta_\delta)$.

(iii) Each narrow mandate permits r with probability $\psi_r = \eta + (1-\eta)\rho_r$ and each $j \ne r$ with probability $\psi_j = (1-\eta)\rho_j$, independently, so ψ is deterministic and, for $\eta > 0$, $\rho_j/\psi_j > 1$ for $j \ne r$, which makes $Y = \rho_r/(\eta + (1-\eta)\rho_r) < 1$. The argument of (ii) shows that the probability of financing tends to one if $1-f < Y$ and to zero if $1-f > Y$, so $f_n(\delta) \to 1 - Y = \eta(1-\rho_r)/(\eta + (1-\eta)\rho_r)$ for every $\delta \in (0,1)$.

(iv) With blocks of k, $n_A = kX$, where X is binomial with $\bar{n}/k$ trials and probability $\frac{1}{2}$, so $2n_A - \bar{n} = k(2X - \bar{n}/k)$ has mean zero and variance $k\bar{n}$, and the argument of (i) with $\sqrt{k\bar{n}}$ in place of $\sqrt{\bar{n}}$ gives $\sqrt{n}\, f_n(\delta) \to z_{1-\delta/2}\sqrt{k}$. □

Theorem 19

(i) In any assignment that meets every requirement, at most K_j of the n_j investors favoring j are placed in it, so at least D_j of them are moved, each at his own cost, and the total cost is at least C_n. The same holds for a fractional assignment $y \ge 0$ with $\sum_j y_{ij} = 1$ and $\sum_i y_{ij} = K_j$, whose cost is $\sum_i c_i(1 - y_{ij(i)})$: the weights $w_i = 1 - y_{ij(i)} \in [0,1]$ of the investors favoring j sum to at least D_j, and $\sum_i c_i w_i$ over them is least, subject to that, when the weight falls on the D_j smallest costs. Because $\sum_j n_j = \sum_j K_j$, the investors moved off over-subscribed projects, $\sum_j D_j$ in number, equal the vacant places $\sum_j (K_j - n_j)_+$, and an investor's cost does not depend on which other project receives his unit, so moving the D_j cheapest supporters of each over-subscribed project, whose costs are finite by hypothesis, attains C_n. Pro rata exposure is the fractional assignment $y_{ij} = \rho_j$, which meets every requirement because $n\rho_j = K_j$, and it costs $\sum_i (1 - \rho_{j(i)})c_i \ge C_n$.

(ii) Investor i's value of an assignment is $-c_i$ if his unit is placed outside $j(i)$ and zero otherwise, so the assignment of (i) maximizes the sum of reported values, and the Vickrey–Clarke–Groves mechanism, which charges each investor the loss his presence imposes on the others' reported values, the others being assigned efficiently to the places with one left vacant as in the assignment game of Leonard (1983), makes truthful reporting of $(j(i), c_i)$ a dominant strategy. With two projects, let $w_i = c_i$ if i favors A and $w_i = -c_i$ if he favors B, his value of a place in A relative to one in B, and write $w_{(l)}$ for the l-th largest report. The efficient assignment gives the K_A places in A to the investors with the K_A largest reports. Without a successful investor, the others would fill his place with the investor of the $(K_A + 1)$-th largest report or leave it vacant, whichever is worth more, so his Clarke payment is $\max\{0, w_{(K_A+1)}\}$; without an unsuccessful investor, they would keep the same investors in A or move the one with the K_A-th largest report to the vacant place in B, so his payment is $\max\{0, -w_{(K_A)}\}$. When A is over-subscribed, the $K_A + 1$ largest reports belong to investors favoring A, so $w_{(K_A)} \geq 0$, places in B are free, and the price of A is $w_{(K_A+1)}$, the largest cost among the D_A investors moved. When neither project is over-subscribed, $w_{(K_A)} \geq 0 \geq w_{(K_A+1)}$ and nobody pays; when B is over-subscribed the roles of the projects are exchanged.

(iii) Call an investor's favored project and his flexibility his *label*, and let N_j be the number of flexible investors favoring j. Conditionally on the labels, the costs of those N_j investors are independent draws from G_c, independent of n_j and D_j, and the program can be financed exactly when $D_j \leq N_j$ for every j. Let $\mathcal{G}_n$ be the event that $N_j \geq \varphi\rho_j n/2$ and $D_j \leq (\varphi\rho_j n/2)^{2/3}$ for every j; by Hoeffding's inequality its complement has probability exponentially small in $n^{1/3}$, so the probability of financing tends to one. Given the labels and $D \leq N$, the expected sum of the D smallest of N draws is at most D/N times the expected sum of all of them, that is, $D\,\mathbb{E}[c]$, so the expected least cost on the event of financing is at most $\mathbb{E}[c]\sum_j \mathbb{E}[D_j] \leq \mathbb{E}[c]\sum_j \sqrt{n\rho_j(1-\rho_j)}$, because $\mathbb{E}[D_j] \leq \mathbb{E}|n_j - K_j| \leq (\operatorname{Var} n_j)^{1/2}$.

Now let $g_c(0) > 0$. Write $e(l, N)$ for the expected l-th smallest of N draws, and represent that draw as $G_c^{-1}(U_{(l)})$, where $U_{(l)}$ is the l-th smallest of N uniform draws and has mean $l/(N + 1)$. Because g_c is positive and continuous at zero, for each $\epsilon > 0$ there is $u_\epsilon > 0$ with $|G_c^{-1}(u) - u/g_c(0)| \leq \epsilon u$ for $u \leq u_\epsilon$. The event $U_{(l)} > u_\epsilon$, on which fewer than l of the N draws fall at or below $G_c^{-1}(u_\epsilon)$, so that for each draw fewer than l of the other $N-1$ do, contributes at most $N\,\mathbb{E}[c]\,\mathbb{P}(\mathrm{Bin}(N-1, u_\epsilon) < l)$ to $e(l, N)$, which is exponentially small in N uniformly in $l \leq N^{2/3}$. Hence

$$\sum_{l \leq D} e(l, N) \;=\; \frac{D(D+1)}{2(N+1)\,g_c(0)}\,(1 + O(\epsilon)) + o(1)$$

uniformly in $D \leq N^{2/3}$ as N grows. On the complement of $\mathcal{G}_n$ the least cost, where it is defined, is at most the sum of all costs, whose conditional expectation is at most $n\,\mathbb{E}[c]$, so that event contributes nothing in the limit. By the central limit theorem $(n_j - K_j)/\sqrt{n}$ converges in distribution to a normal variable with mean zero and variance $\rho_j(1-\rho_j)$, and its square is uniformly integrable because its fourth moment is bounded, so $\mathbb{E}[D_j^2]/n \to \rho_j(1-\rho_j)/2$, the second moment of the positive part of that variable, while $\mathbb{E}[D_j]/n \to 0$ and $N_j/n \to \varphi\rho_j$, with $n/N_j \leq 2/(\varphi\rho_j)$ on

$\mathcal{G}_n$. The expected cost of moving j's supporters on the event of financing therefore converges to $(1-\rho_j)/(4\varphi g_c(0))$ up to a factor $1+O(\epsilon)$, and since ϵ is arbitrary and $\sum_j(1-\rho_j) = m-1$, the expected least cost on that event converges to $(m-1)/(4\varphi g_c(0))$. With two projects, ϖ_n is zero or the D-th smallest cost among the $N \geq \varphi\rho_{\min} n/2$ flexible supporters of the over-subscribed project, with $D = O_p(\sqrt{n})$; $U_{(D)} = O_p(D/N)$ by Markov's inequality, and $G_c^{-1}(u) \leq 2u/g_c(0)$ for small u, so $\sqrt{n}\,\varpi_n = O_p(1)$. If every investor is flexible, the expected cost of pro rata exposure is $\sum_i \mathbb{E}[(1-\rho_{j(i)})c_i] = n\,\mathbb{E}[c]\sum_j \rho_j(1-\rho_j)$ by independence.

(iv) Conditionally on ψ, $n_j/n \to \psi_j$ almost surely, so $D_j/n_j \to u_j = (1-\rho_j/\psi_j)_+$ when $\psi_j > 0$, while $D_j = 0$ for all large n when $\psi_j = 0$. Let $\bar{c}_N(u)$ be the sum of the $\lfloor uN \rfloor$ smallest of N draws, divided by N. For $\epsilon > 0$, the strong law applied to the indicators $\mathbf{1}\{c \leq G_c^{-1}(u \pm \epsilon)\}$ shows that, for all large N, the $\lfloor uN \rfloor$ smallest draws include every draw below $G_c^{-1}(u-\epsilon)$ and none above $G_c^{-1}(u+\epsilon)$, so that, applying the strong law to $c\,\mathbf{1}\{c \leq G_c^{-1}(u \pm \epsilon)\}$, the limit points of $\bar{c}_N(u)$ lie between $\int_0^{u-\epsilon} G_c^{-1}$ and $\int_0^{u+\epsilon} G_c^{-1}$; letting $\epsilon \to 0$, $\bar{c}_N(u) \to \int_0^u G_c^{-1}$, and because $\bar{c}_N$ is nondecreasing in u and the limit is continuous, the convergence holds along $D_j/n_j \to u_j$. Hence $C_n/n \to \sum_j \psi_j \int_0^{u_j} G_c^{-1}(u)\,du$ almost surely. Because $0 \leq C_n/n \leq n^{-1}\sum_i c_i$, which converges in mean, the convergence holds in expectation. The limit is positive when $\psi \neq \rho$, since then some $\psi_j > \rho_j$, $u_j > 0$, and G_c^{-1} is positive on $(G_c(0), 1) = (0, 1)$.

(v) With two equal projects, n_A is binomial with n trials and probability $\frac{1}{2}$, so that $D_n/\sqrt{n} = |n_A - n/2|/\sqrt{n} \Rightarrow |Z|/2$ by the central limit theorem and $D_n \to \infty$ in probability. Let N be the number of flexible supporters of the over-subscribed project, so that $N/n \to \varphi/2$ almost surely, and, conditionally on the labels, represent their costs as $G_c^{-1}(U_{(l)})$ for the order statistics of N uniform variables, written $U_{(l)} = \Theta_l/\Theta_{N+1}$, where Θ_l is the sum of the first l of a sequence of independent standard exponential variables. Since $\Theta_D/D \to 1$ and $\Theta_{N+1}/N \to 1$ in probability, $NU_{(D)}/D \to 1$, and since $G_c^{-1}(u)/u \to 1/g_c(0)$ as $u \to 0$ and $U_{(D)} \to 0$, the price $\varpi_n = G_c^{-1}(U_{(D)})$ satisfies $N\varpi_n/D \to 1/g_c(0)$, so that $\sqrt{n}\,\varpi_n = (D_n/\sqrt{n})(n/N)(N\varpi_n/D_n) \Rightarrow |Z|/(\varphi g_c(0))$. For the cost, $\sum_{l \leq D}\Theta_l = \sum_{i \leq D}(D-i+1)E_i$, where the E_i are the exponential variables, has mean $D(D+1)/2$ and variance $\sum_{i\leq D}(D-i+1)^2 = O(D^3)$, so that $2N\sum_{l\leq D} U_{(l)}/D^2 \to 1$ in probability; because $G_c^{-1}(u) = u(1+o(1))/g_c(0)$ uniformly on $[0, U_{(D)}]$, $2Ng_c(0)C_n/D_n^2 \to 1$, and $C_n = (D_n^2/n)(n/(2N))(1+o_p(1))/g_c(0) \Rightarrow Z^2/(4\varphi g_c(0))$. Each of the three variables is a continuous function of $D_n/\sqrt{n}$ multiplied by factors that converge in probability, so the convergence is joint. For the expectation, let $\mathcal{E}_n$ be the event that $D_n \leq N^{2/3}$, $N \geq \varphi n/4$ and $U_{(D)} \leq u_\epsilon$, where u_ϵ is as in the proof of (iii) with $\epsilon \leq 1/g_c(0)$. On $\mathcal{E}_n$ the program can be financed and $\varpi_n \leq 2U_{(D)}/g_c(0)$, and $n\,\mathbb{E}[U_{(D)}^2 \mid D, N] = n\,D(D+1)/((N+1)(N+2))$ has bounded expectation, so that $\sqrt{n}\,\varpi_n \mathbf{1}_{\mathcal{E}_n}$ has a bounded second moment and is uniformly integrable; since $\mathbb{P}(\mathcal{E}_n) \to 1$, it converges in distribution to $|Z|/(\varphi g_c(0))$, and its expectation converges to $\mathbb{E}|Z|/(\varphi g_c(0))$. On the event of financing outside $\mathcal{E}_n$, ϖ_n is at most the sum of the costs of the N flexible supporters of the over-subscribed project. The first two conditions defining $\mathcal{E}_n$ depend only on the labels, they fail with probability exponentially small in a power of n, and given the labels the expected sum is at most $n\,\mathbb{E}[c]$. When they hold and $U_{(D)} > u_\epsilon$, each supporter has fewer than D of the other $N-1$ costs at or below $G_c^{-1}(u_\epsilon)$, an event independent of his own cost, so the expected sum on that event is

at most $N\,\mathbb{E}[c]\,\mathbb{P}(\mathrm{Bin}(N-1,u_\epsilon)<D)$, which is also exponentially small in a power of n because $D \le N^{2/3}$. Hence $\sqrt{n}\,\mathbb{E}[\varpi_n \mathbf{1}_{\mathcal{E}_n^c}]$ over the event of financing tends to zero, and conditionally on the event of financing, whose probability tends to one, $\sqrt{n}\,\mathbb{E}[\varpi_n] \to \mathbb{E}|Z|/(\varphi g_c(0))$. Finally, let costs be uniform on $[0,1]$, every investor flexible and $n = 2k$. Given $D = d$, the over-subscribed project has $k+d$ supporters, and the expected sum of the d smallest of $k+d$ uniform variables is $\sum_{l\le d} l/(k+d+1) = d(d+1)/(2(k+d+1))$. Since $\mathbb{P}(D=d) = 2\binom{2k}{k+d}4^{-k}$ for $d \ge 1$ and $\binom{2k}{k+d}/(k+d+1) = \binom{2k+1}{k+d+1}/(2k+1)$,

$$\mathbb{E}[C_n] = \frac{1}{(2k+1)4^k}\sum_{j\ge k+1}\binom{2k+1}{j}(j-k-1)(j-k).$$

With X binomial with $2k+1$ trials and probability $\frac{1}{2}$, $(j-k-1)(j-k) = (j-k-\frac{1}{2})^2 - \frac{1}{4}$, and because X is symmetric about $k+\frac{1}{2}$ the sum over $j \ge k+1$ is half the sum over all j, which is $\frac{1}{2}\,2^{2k+1}(\mathrm{Var}\,X - \frac{1}{4}) = 4^k k/2$. Hence $\mathbb{E}[C_n] = k/(2(2k+1)) = n/(4(n+1))$. □

Proposition 20

Under the mechanism that pays investors to move, consider an investor who is indifferent between the projects, $c_i = 0$. If he declares A, he is, with probability one, the cheapest of A's supporters, since others' costs have no atoms; because the others alone over-subscribe A, at least two investors are moved off it, he is one of them, and he is paid the cost of the marginal investor moved, which, whether it is taken to be the largest cost among those moved or the smallest among those who stay, is another investor's cost and positive with probability one. If he declares B, then $n_B \le n - K_A - 1 < K_B$, so B is under-subscribed, he is placed in it, and he is paid nothing. His expected gain from declaring A is therefore positive, and by continuity so is that of an investor who favors B at a small enough cost, who is moved to the project he favors and paid for it. Under the uniform-price assignment, let t be the K_A-th largest report among the others. By the payments in the proof of Theorem 19(ii), investor i receives a place in A exactly when his report w_i exceeds t, and he then pays $\max\{0,t\}$, while in B he pays $\max\{0,-t\}$; his payoff in A less his payoff in B is his true value of A over B less t, which his report does not affect. Reporting truthfully therefore places him in A exactly when that difference is positive, and no misstatement of his favored project or his cost, each of which is a misstatement of w_i, gains. □

Corollary 21

By Theorem 19(iii) the book finances every project with probability tending to one, the mechanisms of part (ii) implement its least-cost assignment, and its expected cost on the event of financing converges to $(m-1)/(4\varphi g_c(0))$. A narrow investor will not accept a share of a project he does not permit, so a blind pool admits only flexible investors, whose capital, about φn, falls short of the requirements n unless $\varphi = 1$; with every investor flexible, pro rata exposure has expected

cost $n\,\mathbb{E}[c]\sum_j \rho_j(1-\rho_j)$. Deal by deal, every project is financed only if $n_j = K_j$ for every j, and $\mathbb{P}(n_1 = K_1)$ is at most the largest point probability of a binomial variable with n trials and probability $\rho_1 \in (0,1)$, which is of order $1/\sqrt{n}$. Under systematic mandates the book's cost divided by n converges in expectation to the positive limit of part (iv), and the pool's to $\mathbb{E}[c]\,\mathbb{E}[\sum_j \psi_j(1-\rho_j)] > 0$, while part (i) shows that the book's never exceeds the pool's. When every investor permits every project, $\mathcal{F}(\Gamma, \mathcal{W})$ depends only on $\mathcal{W}$, so the two arrangements have the same formation structure among the investors willing in both. □

Theorem 22

Reports are truthful in parts (i)–(iii) and (v), and ϖ_n is the price of the over-subscribed project, zero if neither is. Under the uniform-price assignment every investor is flexible, so the price is the largest cost among the investors moved, and a placed investor's burden y_i is ϖ_n if he favors the over-subscribed project and is kept in it, his own cost, at most ϖ_n, if he is moved, and zero otherwise. Write $\varsigma = 0$ if the premium is retained and $\varsigma = \frac{1}{2}$ if it is returned, since a returned premium gives the holder of each place $K\varpi_n/n = \varpi_n/2$, so that a placed investor's net burden is $y_i - \varsigma\varpi_n$, and write $\Xi = \gamma|Z|/2$.

(i) A placed investor who accepts obtains $v_i - (y_i - \varsigma\varpi_n)$ if the book closes and zero otherwise, while one who withdraws obtains zero, so withdrawing exactly when the margin is below the net burden is a best reply to every behavior of the others and weakly dominates every other rule. Conditionally on favored projects and costs, margins are independent of them and of one another, and ties have probability zero. An investor who does not favor the over-subscribed project has net burden $-\varsigma\varpi_n \le 0 < v_i$ and never withdraws. Hence $Q_n = X_n + Y_n$, where X_n counts the K investors kept in the over-subscribed project whose margins are below $(1-\varsigma)\varpi_n$ and Y_n counts the moved investors who withdraw, each of whom has a margin below ϖ_n. Conditionally, X_n is binomial with K trials and probability $G_v((1-\varsigma)\varpi_n)$, and $\mathbb{E}[Y_n \mid \cdot] \le D_n G_v(\varpi_n)$. By Theorem 19(v) with $\varphi = 1$, $\sqrt{n}\,\varpi_n \Rightarrow |Z|/g_c(0)$ and $D_n/\sqrt{n} \Rightarrow |Z|/2$, so that $\varpi_n \to 0$ in probability and $G_v((1-\varsigma)\varpi_n) = g_v(0)(1-\varsigma)\varpi_n(1+o_p(1))$. The conditional mean of X_n, divided by $\sqrt{n}$, is therefore $\frac{1}{2}g_v(0)(1-\varsigma)\sqrt{n}\,\varpi_n(1+o_p(1)) \Rightarrow (1-\varsigma)\Xi$; its conditional variance is at most its conditional mean, of order $\sqrt{n}$, so that $X_n/\sqrt{n}$ less its conditional mean divided by $\sqrt{n}$ tends to zero in probability by Chebyshev's inequality applied conditionally; and $\mathbb{E}[Y_n \mid \cdot]$ is of order one in probability, so that $Y_n/\sqrt{n} \to 0$ in probability by Markov's inequality applied conditionally. Hence $Q_n/\sqrt{n} \Rightarrow (1-\varsigma)\Xi$, which is $\gamma|Z|/2$ or $\gamma|Z|/4$. The limit has a continuous distribution function, positive and below one on $(0,\infty)$. If $S_n/\sqrt{n} \to a > 0$, the probability of closing, $\mathbb{P}(Q_n/\sqrt{n} \le S_n/\sqrt{n})$, converges to $\mathbb{P}((1-\varsigma)\Xi \le a)$; if $a = 0$, it is at most $\mathbb{P}(Q_n/\sqrt{n} \le \epsilon)$ for large n and every $\epsilon > 0$, whose limit $\mathbb{P}((1-\varsigma)\Xi \le \epsilon)$ tends to zero with ϵ. If $S_n/\sqrt{n} \to \infty$ the probability tends to one because $Q_n/\sqrt{n}$ is tight, and otherwise $S_n/\sqrt{n} \le A$ along a subsequence and the probability is at most $\mathbb{P}(Q_n/\sqrt{n} \le A) \to \mathbb{P}((1-\varsigma)\Xi \le A) < 1$ along it.

(ii) Favored projects and costs are realized after enrollment, independently of margins, and the n investors placed are drawn at random among those who enroll, so the profile of the placed

investors' favored projects and costs has the same distribution whatever the enrollment decisions and is independent of the event that the book closes. By exchangeability every placed investor's expected net burden is $v_n^* = \mathbb{E}[\sum_i (y_i - \varsigma\varpi_n)]/n$. An investor who enrolls therefore obtains $\pi_i(v_i - v_n^*)$, where $\pi_i \geq 0$ is the probability, which depends on the others' behavior, that he is placed and the book closes, while declining yields zero, so that enrolling exactly when $v_i \geq v_n^*$ is a best reply to every behavior of the others and weakly dominates every other rule. With the premium retained, $\sum_i y_i = K\varpi_n + C_n$, so $v_n^* = (K\,\mathbb{E}[\varpi_n] + \mathbb{E}[C_n])/n$. By Theorem 19(iii) and (v), $\mathbb{E}[C_n] \to 1/(4g_c(0))$ and $\sqrt{n}\,\mathbb{E}[\varpi_n] \to \mathbb{E}|Z|/g_c(0) = \sqrt{2/\pi}/g_c(0)$, so that $\sqrt{n}\,v_n^* \to 1/(\sqrt{2\pi}\,g_c(0))$. The number declining is binomial with $n + S_n$ trials and probability $G_v(v_n^*) = g_v(0)v_n^*(1 + o(1))$, so that, with $S_n = O(\sqrt{n})$, its mean divided by $\sqrt{n}$ converges to a^* and its variance is at most its mean; hence $Q_n'/\sqrt{n} \to a^*$ in probability, and the closing probabilities follow. The limit in (i) with the premium retained has mean $\frac{1}{2}\gamma\,\mathbb{E}|Z| = \gamma/\sqrt{2\pi} = a^*$.

(iii) With the premium returned, the placed investors' net burdens sum to $K\varpi_n + C_n - n\varpi_n/2 = C_n$, so that $v_n^* = \mathbb{E}[C_n]/n$ and the dominance argument of (ii) applies. Then Q_n' is binomial with $n+S_n$ trials and probability $p_n = G_v(\mathbb{E}[C_n]/n)$, and $(n+S_n)p_n = (1+S_n/n)\,g_v(0)\,\mathbb{E}[C_n](1+o(1)) \to g_v(0)/(4g_c(0)) = \mu_v$ when $S_n/n \to 0$, so that Q_n' converges in distribution to a Poisson variable with mean μ_v, and the closing probabilities for fixed s and at critical load follow. If $S_n \to \infty$, Markov's inequality gives $\mathbb{P}(Q_n' > S_n) \leq (n + S_n)p_n/S_n = np_n/S_n + p_n \to 0$, since np_n is bounded.

(iv) Under binding enrollment the closing is settled before reports are made, so an investor's report affects only his own place and payment and, when the premium is returned, his share of it. With the premium retained, Theorem 19(ii) applies and truthful reporting is dominant. With it returned, truthful reporting still maximizes the value of his place less his payment, by the argument in the proof of Proposition 20, and his share of the premium is $K\varpi_n/n = \frac{1}{2}\max\{0, \hat{w}_{(K+1)}, -\hat{w}_{(K)}\}$, where $\hat{w}_{(l)}$ is the l-th largest of all n reports. With $n \geq 4$, so that $K \geq 2$, whatever he reports, $\hat{w}_{(K+1)}$ lies between $w_{(K+1)}^{-i}$ and $w_{(K)}^{-i}$ and $\hat{w}_{(K)}$ between $w_{(K)}^{-i}$ and $w_{(K-1)}^{-i}$, and since $(x, y) \mapsto \max\{0, x, -y\}$ changes by at most the larger of the changes in its two arguments, his share of the premium differs between any two reports by at most $\frac{1}{2}\max\{w_{(K-1)}^{-i} - w_{(K)}^{-i},\ w_{(K)}^{-i} - w_{(K+1)}^{-i}\}$, which bounds the gain from any misreport. Under a retained veto with the premium retained, let A be over-subscribed. The price is the highest unsuccessful report, and the investor who makes it can lower it toward the next report without changing his place; the price falls, the set of investors kept in A whose margins are below their net burden weakly shrinks, and the probability of closing weakly rises, strictly when a margin lies between the two prices. The gap between the two reports is the spacing between consecutive costs among the supporters of A at $U_{(D)}$, which in the representation of the proof of Theorem 19(v) is $G_c^{-1}(\Theta_D/\Theta_{N+1}) - G_c^{-1}(\Theta_{D-1}/\Theta_{N+1})$, of order $1/n$ in probability, so the conditional mean of X_n changes by a quantity of order one in probability. When $D = 1$ the next report belongs to a supporter of B and the price can fall to zero, but that event has probability tending to zero. With the premium returned, lowering the price also raises the net burden of every investor moved, the deviator included, by half the change, so that the number of withdrawals among those moved can rise and the effect on the probability of closing is not signed in general, although the expected number of withdrawals among those kept in A falls.

(v) Let i favor A with cost c, and let the other $n-1$ investors use a common rule that treats the projects alike and enrolls a share $e(c') \geq \frac{1}{2}$ of the investors of each cost c'. At critical load the book closes exactly when all n enroll, and conditionally on the others enrolling their favored projects are independent and equally likely, independent of their costs, which are independent draws from the distribution $\tilde{G}_c$ with density $\tilde{g}_c = g_c e / \int g_c e$, so that $g_c/2 \leq \tilde{g}_c \leq 2g_c$. Write $\bar{x}(l, N)$ for the expected l-th smallest of N draws from $\tilde{G}_c$, and let M, the number of others favoring A, be binomial with $2K-1$ trials and probability $\frac{1}{2}$. If $M \geq K$, project A is over-subscribed by $d = M+1-K$, the price is the d-th smallest cost among the $M+1$ supporters of A, and, with $X_{(d)}$ the d-th smallest of the others' M costs, i either is kept and pays $X_{(d)}$, when $c > X_{(d)}$, or is moved at cost $c < X_{(d)}$ with a price of at least c; in either case his net burden, his burden less half the price, is at most $X_{(d)}/2$. If $M \leq K-2$, project B is over-subscribed by $d = K-1-M$, its $K+d$ supporters are all others, i pays nothing, and his net burden is minus half the d-th smallest of their costs. With $p_d = \mathbb{P}(M = K+d-1) = \mathbb{P}(M = K-d)$, so that $\mathbb{P}(M = K-1-d) = p_{d+1}$ and $p_{d+1}/p_d = (K-d)/(K+d)$, the expected net burden of i, given his type, is at most

$$\tfrac{1}{2} \sum_{d=1}^{K} \left[p_d \, \bar{x}(d, K+d-1) - p_{d+1} \, \bar{x}(d, K+d) \right].$$

Dropping one of $N+1$ draws at random leaves the l-th smallest of the remaining N equal to the $(l+1)$-th smallest of the $N+1$ when the dropped draw is among the l smallest, which has probability $l/(N+1)$, and to the l-th smallest otherwise, so that $(N+1)\bar{x}(l, N) = l \, \bar{x}(l+1, N+1) + (N+1-l) \, \bar{x}(l, N+1)$. With $N = K+d-1$ and $l = d$, the bracket equals $p_d \, d \, [\bar{x}(d, K+d) + \bar{x}(d+1, K+d)]/(K+d)$, and the bound is at most $\sum_d p_d (d/K) \, \bar{x}(d+1, K+d)$. Because g_c is positive and continuous at zero there is $x_0 > 0$ with $g_c \geq g_c(0)/2$ on $[0, x_0]$, so that $\tilde{G}_c(x) \geq g_c(0)x/4$ there and $\tilde{G}_c^{-1}(u) \leq 4u/g_c(0)$ for $u \leq u_0 = g_c(0)x_0/4$. Representing the l-th smallest of N draws as $\tilde{G}_c^{-1}(U_{(l)})$ with $\mathbb{E}[U_{(l)}] = l/(N+1)$, and bounding it on the event $U_{(l)} > u_0$ by the sum of all N draws, each of which, on that event, has fewer than l of the other $N-1$ draws at or below $\tilde{G}_c^{-1}(u_0)$, an event independent of it, gives

$$\bar{x}(l, N) \;\leq\; \frac{4l}{(N+1)\, g_c(0)} + 2N \, \mathbb{E}[c] \, \mathbb{P}(\mathrm{Bin}(N-1, u_0) < l).$$

The first term contributes at most $(8/(g_c(0)K^2)) \sum_d p_d d^2 \leq 4/(g_c(0)K)$, since $\sum_{d \geq 1} p_d d^2 \leq \mathbb{E}[(K-M)^2] = K/2$. In the second, the binomial probability is at most $e^{-u_0 K/8}$ when $d+1 \leq u_0(K+d-1)/2$, by the Chernoff bound, and the remaining d have total probability at most $e^{-\xi_0 K}$ for some $\xi_0 > 0$ depending on u_0, by Hoeffding's inequality; since $(d/K) \cdot 2(K+d) \, \mathbb{E}[c] \leq 4K \, \mathbb{E}[c]$, the second term contributes at most $4K \, \mathbb{E}[c](e^{-u_0 K/8} + e^{-\xi_0 K})$, which is at most a constant divided by K; for the finitely many small K at which the Hoeffding bound has not yet taken effect, the net burden is in any case at most $4K \, \mathbb{E}[c]$, which the constant absorbs. Hence the expected net burden is at most C_0/n for a constant C_0 that depends only on $g_c(0)$, x_0 and $\mathbb{E}[c]$, and by symmetry the same bound holds for an investor who favors B.

An investor who enrolls obtains, in expectation, the probability that all others enroll, which is at least $2^{1-n} > 0$, times his margin less his expected net burden given his type, and declining yields zero, so in any symmetric equilibrium in such rules every investor whose margin exceeds C_0/n enrolls, and since types are independent the probability that all n enroll is at least $(1 - G_v(C_0/n))^n$, which converges to $e^{-g_v(0)C_0}$ because $nG_v(C_0/n) \to g_v(0)C_0$. For existence, let n be large enough that $G_v(C_0/n) \le \frac{1}{2}$, and let $\mathcal{Y}$ be the set of measurable functions e from costs to $[\frac{1}{2}, 1]$, a convex subset of $L^\infty(G_c)$ that is compact and metrizable in the weak* topology. For $e \in \mathcal{Y}$ let $N_e(c)$ be the expected net burden of an investor with cost c when the others enroll at rate e, and let $\Phi(e)(c) = 1 - G_v(N_e(c))$, the enrollment rate of investors with cost c who enroll exactly when their margins exceed $N_e(c)$, which is a best reply. By the bound, $\Phi(\mathcal{Y}) \subseteq \mathcal{Y}$. $N_e(c)$ is a ratio of integrals of an integrable function of the others' costs against the product measures $\prod_k e(x_k)g_c(x_k)\,dx_k$ and of powers of $\int e g_c$, and if $e_m \to e$ weak* then the products $e_m^{\otimes(n-1)}$ converge weak* to $e^{\otimes(n-1)}$, because they are bounded and converge against products of integrable functions, whose span is dense; hence $N_{e_m}(c) \to N_e(c)$ for every c, $\Phi(e_m) \to \Phi(e)$ pointwise and therefore weak*, and Φ is continuous. By the Schauder–Tychonoff theorem Φ has a fixed point, which is a symmetric equilibrium in rules of the required kind. □

Corollary 23

$S_n(\delta)$ is the least S for which the relevant shortfall is at most S with probability at least $1 - \delta$. Under a retained veto the shortfall Q_n does not depend on S, and $Q_n/\sqrt{n}$ converges in distribution to $\gamma|Z|/2$ or $\gamma|Z|/4$, whose distribution functions are continuous and strictly increasing on $(0, \infty)$; the $(1 - \delta)$-quantile of $|Z|$ is $z_{1-\delta/2}$, so the quantiles converge and $S_n(\delta)/\sqrt{n}$ tends to $z_{1-\delta/2}\gamma/2$ or $z_{1-\delta/2}\gamma/4$. Under binding enrollment the shortfall is binomial with $n + S$ trials, and the probability that it is at most S rises with S, because an additional invitation raises the shortfall by at most one. With the premium retained, Theorem 22(ii) gives, for $S = a\sqrt{n}$, a closing probability tending to one if $a > a^*$ and to zero if $a < a^*$, so that $S_n(\delta) \le a\sqrt{n}$ eventually for every $a > a^*$ and $S_n(\delta) > a\sqrt{n}$ eventually for every $a < a^*$, whence $S_n(\delta)/\sqrt{n} \to a^*$ for every $\delta \in (0, 1)$, and the ratio of the two limits is $z_{1-\delta/2}(\gamma/2)/(\gamma/\sqrt{2\pi}) = z_{1-\delta/2}\sqrt{\pi/2}$. With the premium returned, for each fixed integer s the closing probability converges to $\mathbb{P}(\text{Poisson}(\mu_v) \le s)$ by Theorem 22(iii), and when $1 - \delta$ is not a value of that distribution function, which excludes countably many δ, the least s with closing probability at least $1 - \delta$ converges to the least s with $\mathbb{P}(\text{Poisson}(\mu_v) \le s) \ge 1 - \delta$. □

Proposition 24

(i) Without investor i, the other $n - 1$ investors compete for K places in each project, one of which stays vacant, and the uniform-price assignment prices a project only if more than K of them favor it, at the $(K + 1)$-th largest of their reported values of a place in that project over one in the other. If A is over-subscribed when i is present, at most $K - 1$ investors favor B, so B is not over-subscribed without i, and the $(K + 1)$-th largest of the others' reports is at most the $(K + 1)$-th largest of all

reports, so that $\varpi_n^{-i} \le \varpi_n$; if neither project is over-subscribed, neither is without i, and $\varpi_n^{-i} = 0$; the case of B is symmetric. Hence $0 \le \Lambda_n \le K\varpi_n$. Investor i's payoff is the value of his place less his payment plus $(K/n)\varpi_n^{-i}$, and the last term does not depend on his report. By the argument in the proof of Proposition 20, a truthful report maximizes the first two whatever the others report, so truthful reporting is dominant.

(ii) Every investor is flexible, so the program can always be financed. Let A be over-subscribed by $D \ge 1$, let $N = K + D$ be the number of its supporters, and write $X_{(l)}$ for the l-th smallest of their costs, with $X_{(0)} = 0$. The $K + 1$ highest reports belong to the K investors kept in A and to the investor whose cost $X_{(D)}$ sets the price. Removing one of them leaves A over-subscribed by $D - 1$ among the others, with price $X_{(D-1)}$, which is zero when $D = 1$ because A is then no longer over-subscribed. Removing any other investor leaves the price at $X_{(D)}$, since a supporter of A moved at a cost below $X_{(D)}$ takes one of the D smallest costs with him, and a supporter of B does not change A's over-subscription. Hence

$$\Lambda_n = \frac{K(K+1)}{n}\,(X_{(D)} - X_{(D-1)}),$$

and the case of B is symmetric. In the representation of the proof of Theorem 19(v), $U_{(D)} - U_{(D-1)} = E_D/\Theta_{N+1}$, where E_D is the D-th of the independent standard exponential variables, so that $N(U_{(D)} - U_{(D-1)})$ converges in distribution to a standard exponential variable independent of the labels. D_n is a function of the labels and, given them, E_D is standard exponential, so E_D is independent of the labels; every other factor converges in probability to a constant, and Slutsky's lemma gives joint convergence with the exponential limit independent of Z. Because g_c is positive and continuous at zero, for each $\epsilon \in (0,1)$ there is $x_\epsilon > 0$ with $|G_c(x) - G_c(y) - g_c(0)(x-y)| \le \epsilon g_c(0)|x-y|$ for $x, y \in [0, x_\epsilon]$. Since $X_{(D)} \to 0$ in probability, $g_c(0)(X_{(D)} - X_{(D-1)})/(U_{(D)} - U_{(D-1)}) \to 1$ in probability, and since $K(K+1)/(nN) \to \frac{1}{2}$ in probability, Λ_n converges in distribution to $E/(2g_c(0))$, with E standard exponential.

For the expectation, redefine $\mathcal{E}_n$ as the event that $D \le N^{2/3}$ and $X_{(D)} \le x_{1/2}$; since $N = K + D \ge n/2$, the bounds below hold on it. On $\mathcal{E}_n$, $X_{(D)} - X_{(D-1)} \le 2(U_{(D)} - U_{(D-1)})/g_c(0)$, and $\mathbb{E}[(U_{(D)} - U_{(D-1)})^2 \mid D, N] = 2/((N+1)(N+2))$, so that $\mathbb{E}[\Lambda_n^2 \mathbf{1}_{\mathcal{E}_n}]$ is bounded and $\Lambda_n \mathbf{1}_{\mathcal{E}_n}$ is uniformly integrable. Off $\mathcal{E}_n$, $\Lambda_n \le nX_{(D)}$, which is at most n times the sum of the supporters' costs. The first condition defining $\mathcal{E}_n$ depends only on the labels and fails with probability exponentially small in a power of n, and given the labels the expected sum is at most $n\,\mathbb{E}[c]$. When it holds and $X_{(D)} > x_{1/2}$, each supporter has fewer than D of the other $N - 1$ costs at or below $x_{1/2}$, an event independent of his own cost, so the expectation of n times the sum on that event is at most $nN\,\mathbb{E}[c]\,\mathbb{P}(\mathrm{Bin}(N-1, G_c(x_{1/2})) < D)$, which is also exponentially small in a power of n. Hence $\mathbb{E}[\Lambda_n \mathbf{1}_{\mathcal{E}_n^c}] \to 0$ and $\mathbb{E}[\Lambda_n] \to 1/(2g_c(0))$.

(iii) With truthful reports, the placed investors' net burdens sum to $K\varpi_n + C_n - \sum_i (K/n)\varpi_n^{-i} = C_n + \Lambda_n$. By exchangeability, as in the proof of Theorem 22(ii), each placed investor's expected net burden is $\mathbb{E}[C_n + \Lambda_n]/n$, and enrolling exactly when the margin is at least that amount is a best reply to every enrollment behavior of the others. Truthful reporting is dominant by (i), so no investor

gains by deviating at either stage. By Theorem 19(iii) and part (ii), $\mathbb{E}[C_n + \Lambda_n] \to 3/(4g_c(0))$, and the argument of Theorem 22(iii) gives the Poisson limit with mean $3g_v(0)/(4g_c(0)) = 3\gamma/4$ and the closing probabilities.

(iv) In the proof of Theorem 22(v), let investor i favor A with cost c. Write $X_{(l)}$ for the l-th smallest of the costs of the other M supporters of A, with $X_{(0)} = 0$. When A is over-subscribed by d, so that $M = K + d - 1$, removing i leaves A over-subscribed by $d - 1$ among the others. His rebate is therefore $\frac{1}{2}X_{(d-1)}$ whatever his cost, while his burden is at most $X_{(d)}$, and his net burden is at most $\frac{1}{2}X_{(d)} + \frac{1}{2}(X_{(d)} - X_{(d-1)})$. When B is over-subscribed, removing i does not change B's price, and his rebate is as before. The bound of that proof therefore rises by at most $\frac{1}{2}\sum_d p_d(\bar{x}(d, K + d - 1) - \bar{x}(d - 1, K + d - 1))$. Since $\tilde{g}_c \geq g_c(0)/4$ on $[0, x_0]$, $\tilde{G}_c^{-1}$ is $4/g_c(0)$-Lipschitz on $[0, u_0]$, so each difference is at most $4/(g_c(0)(K + d))$ plus a term bounded by $2(K + d)\,\mathbb{E}[c]\,\mathbb{P}(\mathrm{Bin}(K + d - 2, u_0) < d)$, exactly as there. The increase is therefore at most a constant divided by n. The equilibrium and existence arguments are unchanged, and truthful reports are dominant by (i). □

Corollary 41

By Theorem 19(v) with $\varphi = 1$ and Proposition 24(ii), (C_n, Λ_n) converges in distribution to $(Z^2/(4g_c(0)), E/(2g_c(0)))$, where E is a standard exponential variable independent of Z, so that $C_n + \Lambda_n$ converges in distribution to $(Z^2 + 2E)/(4g_c(0))$. Z^2 is chi-squared with one degree of freedom and $2E$ with two, and they are independent, so that their sum is chi-squared with three. Every investor is flexible, so the program can always be financed, and $\mathbb{E}[C_n] \to 1/(4g_c(0))$ by Theorem 19(iii), while $\mathbb{E}[\Lambda_n] \to 1/(2g_c(0))$ by Proposition 24(ii). With the premium returned in equal shares the net burdens sum to C_n, as in the proof of Theorem 22(iii). □

Proposition 25

Replacement capital fills every vacated place, so a placed investor's decision to accept affects only his own payoff, and accepting exactly when his margin is at least his net burden, his cost of the place assigned plus his transfer, called his burden below, is optimal whatever the others do. Every investor is flexible, and reports are truthful in parts (i) and (ii).

The rate κ_v. The ratio $G_v(x)/(x(1 - G_v(x)))$ tends to $g_v(0)$ as x falls to zero, so $\kappa_v \leq g_v(0)$, and on $(0, x']$ it is at least $g_v(0)/2$ for some $x' > 0$. If $x(1 - G_v(x))$ is bounded, the ratio is at least $G_v(x')/\sup_x x(1 - G_v(x))$ on $[x', \infty)$, so $\kappa_v > 0$; if margins have a finite mean, $x(1 - G_v(x)) \leq \mathbb{E}[v]$ by Markov's inequality. If $x(1 - G_v(x))$ is unbounded, the ratio is at most $1/(x(1 - G_v(x)))$, which has infimum zero. With margins uniform on $(0, b)$ the ratio is $1/(b - x) \geq 1/b = g_v(0)$, and with margins exponential with rate r it is $(e^{rx} - 1)/x \geq r = g_v(0)$.

The price an investor faces. Let M be the number of investors other than i who favor A, binomial with $n - 1$ trials and probability $\frac{1}{2}$. If $M \geq K$, placing i in A rather than B requires one more of the others who favor A to be moved, so that ϖ_i° is the d-th smallest of their M costs, with

$d = M - K + 1$; if $M \le K - 1$, it spares one of the others who favor B from being moved, and ϖ_i° is minus the d-th smallest of their $n - 1 - M$ costs, with $d = K - M$. In both cases $|\varpi_i^\circ|$ is the d-th smallest of $N = K - 1 + d$ independent draws from G_c, with $d - \frac{1}{2} = |M - \frac{n-1}{2}|$, and the sign of ϖ_i° is independent of $|\varpi_i^\circ|$ and equally likely to be either. Since $d/\sqrt{n} \Rightarrow |Z|/2$, the argument of the proof of Theorem 19(v) gives $\sqrt{n}\,|\varpi_i^\circ| \Rightarrow |Z|/g_c(0)$, so that $Z_n = \sqrt{n}\,g_c(0)\,\varpi_i^\circ$ converges in distribution to a standard normal variable. Choose $x_1 > 0$ with $g_c(0)/2 \le g_c \le 2g_c(0)$ and $g_v(0)/2 \le g_v \le 2g_v(0)$ on $[0, x_1]$, and let $\mathcal{H}_n$ be the event that $d \le N^{2/3}$ and $|\varpi_i^\circ| \le x_1$. On $\mathcal{H}_n$, $|\varpi_i^\circ| \le 2U_{(d)}/g_c(0)$, where $U_{(d)}$ is the d-th smallest of N uniform variables, and $\mathbb{E}[U_{(d)}^4 \mid d] \le ((d+3)/N)^4$, so that $\mathbb{E}[Z_n^4 \mathbf{1}_{\mathcal{H}_n}]$ is bounded, because $\mathbb{E}[d^4] = O(n^2)$ and $N \ge n/2 - 1$. Off $\mathcal{H}_n$, the argument of that proof shows that $\mathbb{E}[|\varpi_i^\circ| \mathbf{1}_{\mathcal{H}_n^c}]$ is exponentially small in a power of n. Hence $Z_n \mathbf{1}_{\mathcal{H}_n}$ and its square are uniformly integrable, and $n^k\,\mathbb{E}[|\varpi_i^\circ| \mathbf{1}_{\mathcal{H}_n^c}] \to 0$ for every k.

(i) Fix i and the others' reports, and let $\varpi_i^\circ > 0$; the case $\varpi_i^\circ < 0$ exchanges the projects, and $\varpi_i^\circ \ne 0$ with probability one. Let w be i's reported value of a place in A over one in B. The assignment places him in A if $w > \varpi_i^\circ$ and in B if $w < \varpi_i^\circ$. If two reports that lead to the same project carried different transfers, an investor whose true value is the report with the larger transfer would gain by making the other, so the transfer is some τ_A for every report above ϖ_i° and some τ_B for every report below. An investor whose value is w bears, before transfers, $w^- = \max\{-w, 0\}$ in A and $w^+ = \max\{w, 0\}$ in B, and dominance applies to every favored project and every cost, so that every real w is a possible value. A truthful report must be no worse than one that changes his project, so that $w^- + \tau_A \le w^+ + \tau_B$, that is $\tau_A - \tau_B \le w$, for every $w > \varpi_i^\circ$, and $\tau_A - \tau_B \ge w$ for every $w < \varpi_i^\circ$; hence $\tau_A - \tau_B = \varpi_i^\circ$. With truthful reports, an investor who favors B is placed in B and bears τ_B; one who favors A with cost $c > \varpi_i^\circ$ is placed in A and bears $\tau_B + \varpi_i^\circ$; and one who favors A with cost $c < \varpi_i^\circ$ is moved and bears $c + \tau_B$, his transfer being τ_B. In every case his transfer is at most his burden.

If $\kappa_v = 0$ the bound is trivial. Otherwise let $\lambda \in (0, g_v(0)/2]$ with $\lambda < \kappa_v$, and write τ_i for i's transfer. Since $\Delta_n \ge -T_n$ and transfers have finite expectations,

$$\mathbb{E}[Q_n] + \lambda\,\mathbb{E}[\Delta_n] \;\ge\; \mathbb{E}[Q_n] - \lambda\,\mathbb{E}\Big[\sum_i \tau_i \mathbf{1}\{i \text{ accepts}\}\Big] = \sum_i \mathbb{E}[\mathbf{1}\{i \text{ withdraws}\} - \lambda\tau_i \mathbf{1}\{i \text{ accepts}\}],$$

and when the mechanism is unsubsidized in expectation the left side may be replaced by $\mathbb{E}[Q_n]$, since then $-\lambda\,\mathbb{E}[T_n] \le 0$. Write $\mathcal{B}_n$ for the right side. Given the others' reports, i's favored project, cost and margin are independent of them, and he withdraws with probability $G_v(b_i^+)$, where b_i is his burden. The term for i is therefore the expectation of $\mathcal{L}(b_i, \tau_i)$, where $\mathcal{L}(x, \tau) = G_v(x^+) - \lambda\tau(1 - G_v(x^+))$. Since $\tau_i \le b_i$, and $\mathcal{L}(x, \tau) \ge \underline{\mathcal{L}}(x)$ whenever $\tau \le x$, where $\underline{\mathcal{L}}(x) = \lambda|x|$ for $x \le 0$ and $\underline{\mathcal{L}}(x) = G_v(x) - \lambda x(1 - G_v(x))$ for $x > 0$, and for $x > 0$ the definition of κ_v gives $x(1 - G_v(x)) \le G_v(x)/\kappa_v$, so that $\underline{\mathcal{L}}(x) \ge (1 - \lambda/\kappa_v)G_v(x) \ge 0$. Omitting the investors who favor A with costs below ϖ_i°, whose terms are nonnegative, the conditional expectation of $\mathcal{L}$ given the others' reports is at least

$$\mathcal{L}_i = \tfrac{1}{2}\underline{\mathcal{L}}(\tau_B) + \tfrac{1}{2}(1 - G_c(\varpi_i^\circ))\underline{\mathcal{L}}(\tau_B + \varpi_i^\circ).$$

For $\epsilon > 0$ let $\upsilon_\epsilon = \min\{1, \inf_{0<x\le\epsilon} \underline{\mathcal{L}}(x)/(\lambda x)\}$ and $\ell_\epsilon = \inf_{x\ge\epsilon} \underline{\mathcal{L}}(x) \ge (1-\lambda/\kappa_v)G_v(\epsilon) > 0$. As x falls to zero, $\underline{\mathcal{L}}(x)/(\lambda x) \to (g_v(0)-\lambda)/\lambda \ge 1$, so that $\upsilon_\epsilon \to 1$ as $\epsilon \to 0$. Let $\varpi_i^\circ < \epsilon$. If $\tau_B \le -\varpi_i^\circ$, then $\mathcal{L}_i \ge \frac{1}{2}\lambda|\tau_B| \ge \frac{1}{2}\lambda\varpi_i^\circ$. If $-\varpi_i^\circ < \tau_B \le 0$, then $\mathcal{L}_i \ge \frac{1}{2}\lambda|\tau_B| + \frac{1}{2}(1-G_c(\varpi_i^\circ))\lambda(\tau_B+\varpi_i^\circ)\upsilon_\epsilon \ge \frac{1}{2}\lambda\varpi_i^\circ(1-G_c(\varpi_i^\circ))\upsilon_\epsilon$. If $\tau_B > 0$ and $\tau_B+\varpi_i^\circ \le \epsilon$, then $\mathcal{L}_i \ge \frac{1}{2}(1-G_c(\varpi_i^\circ))\lambda(\tau_B+\varpi_i^\circ)\upsilon_\epsilon \ge \frac{1}{2}\lambda\varpi_i^\circ(1-G_c(\varpi_i^\circ))\upsilon_\epsilon$, and if $\tau_B + \varpi_i^\circ > \epsilon$, then $\mathcal{L}_i \ge \frac{1}{2}(1-G_c(\varpi_i^\circ))\ell_\epsilon$. In every case

$$\mathcal{L}_i \;\ge\; \tfrac{1}{2}\lambda\varpi_i^\circ(1-G_c(\varpi_i^\circ))\min\{\upsilon_\epsilon,\, \ell_\epsilon/(\lambda\varpi_i^\circ)\},$$

and $\mathcal{L}_i \ge 0$ when $\varpi_i^\circ \ge \epsilon$. By symmetry the same holds with $|\varpi_i^\circ|$ in place of ϖ_i° when $\varpi_i^\circ < 0$, and by exchangeability $\mathcal{B}_n \ge \frac{1}{2}\lambda n\,\mathbb{E}[\mathcal{V}_n]$, where $\mathcal{V}_n = |\varpi_1^\circ|(1-G_c(|\varpi_1^\circ|))\min\{\upsilon_\epsilon, \ell_\epsilon/(\lambda|\varpi_1^\circ|)\}\mathbf{1}\{|\varpi_1^\circ| < \epsilon\}$. Since $\sqrt{n}\,|\varpi_1^\circ| \Rightarrow |Z|/g_c(0)$ and $|\varpi_1^\circ| \to 0$ in probability, the nonnegative variables $\sqrt{n}\,\mathcal{V}_n$ converge in distribution to $\upsilon_\epsilon|Z|/g_c(0)$, and Fatou's lemma gives $\liminf \mathcal{B}_n/\sqrt{n} \ge \frac{1}{2}\lambda\,\upsilon_\epsilon\,\mathbb{E}|Z|/g_c(0)$. Letting $\epsilon \to 0$ gives $\liminf(\mathbb{E}[Q_n] + \lambda\,\mathbb{E}[\Delta_n])/\sqrt{n} \ge \frac{1}{2}\lambda\,\mathbb{E}|Z|/g_c(0)$ for every $\lambda < \lambda_v$, and since the left side is nondecreasing in λ the same holds with λ_v in place of λ on the left; letting λ rise to λ_v on the right gives the bound, since $\frac{1}{2}\min\{\kappa_v, g_v(0)/2\}\,\mathbb{E}|Z|/g_c(0) = \min\{2\kappa_v/g_v(0), 1\}\,\gamma\,\mathbb{E}|Z|/4$.

For the rebate of Proposition 24 less τ_0/n per place, truthful reporting is dominant in the assignment by Proposition 24(i), since the fee depends on no report. Let Q_n^0 be the shortfall when the premium is returned in equal shares, so that $Q_n^0/\sqrt{n} \Rightarrow \gamma|Z|/4$ by Theorem 22(i). Under equal shares every burden is at most $\varpi_n/2$, and under the rebate each investor's burden exceeds it by $\Lambda_{n,i} + \tau_0/n$, where $\Lambda_{n,i} = (K/n)(\varpi_n - \varpi_n^{-i}) \in [0, \varpi_n/2]$ and $\sum_i \Lambda_{n,i} = \Lambda_n$, so that $Q_n \ge Q_n^0$. On the event that $\varpi_n \le x_1/2$, and for $n \ge 2\tau_0/x_1$, every burden under either rule is at most x_1, and given costs and favored projects the expected number of investors who withdraw under the rebate but not under equal shares is at most $\sum_i 2g_v(0)(\Lambda_{n,i} + \tau_0/n) = 2g_v(0)(\Lambda_n + \tau_0)$; off that event it is at most n, and $n\mathbb{P}(\varpi_n > x_1/2) \to 0$ by the estimates in the proof of Theorem 19(v). Hence $\mathbb{E}[Q_n - Q_n^0] \le 2g_v(0)(\mathbb{E}[\Lambda_n] + \tau_0) + o(1)$, which is bounded, so that $Q_n/\sqrt{n}$ and $Q_n^0/\sqrt{n}$ have the same limit in distribution, and their expectations the same limit if either has one. Given costs and favored projects, Q_n^0 is at most a binomial variable with n trials and probability $G_v(\varpi_n)$, so that $\mathbb{E}[(Q_n^0)^2]/n \le \mathbb{E}[nG_v(\varpi_n)^2 + G_v(\varpi_n)]$, which is at most $4g_v(0)^2\Upsilon + 2g_v(0)\,\mathbb{E}[\varpi_n] + (n+1)\mathbb{P}(\varpi_n > x_1/2)$, where $\Upsilon = \sup_n n\,\mathbb{E}[\varpi_n^2\mathbf{1}\{\varpi_n \le x_1/2\}]$ is finite by the same estimates; hence $Q_n^0/\sqrt{n}$ is uniformly integrable and $\mathbb{E}[Q_n]/\sqrt{n} \to \gamma\,\mathbb{E}|Z|/4$. The same bound applies when the premium is retained, since every burden is then at most ϖ_n, so that $\mathbb{E}[Q_n]/\sqrt{n} \to \gamma\,\mathbb{E}|Z|/2$ for the uniform-price assignment that retains its premium, twice the limit under the rebate. For the budget, the transfers of all placed investors sum to $\Lambda_n + \tau_0$, and those of the investors who withdraw are at most their burdens, each at most $\varpi_n + \tau_0/n$, so that the transfers collected from the investors who accept are at least $\tau_0(1 - Q_n/n) - Q_n\varpi_n$. Given costs and favored projects, Q_n is at most a binomial variable with n trials and probability $G_v(\varpi_n + \tau_0/n)$, so that, for $n \ge 2\tau_0/x_1$, $\mathbb{E}[Q_n\varpi_n] \le 2g_v(0)(\Upsilon + \tau_0\,\mathbb{E}[\varpi_n]) + n\,\mathbb{E}[\varpi_n\mathbf{1}\{\varpi_n > x_1/2\}]$, whose last term tends to zero. Since $\mathbb{E}[Q_n]/n \to 0$ and $\mathbb{E}[\varpi_n] \to 0$, the expected transfers collected are at least $\tau_0 - 2g_v(0)\Upsilon - o(1)$, which is positive for all large n when $\tau_0 > 2g_v(0)\Upsilon$. Finally, $-T_n$ is at most the sum of the transfers

of the investors who withdraw, each at most $\varpi_n + \tau_0/n$, so that $\Delta_n \le Q_n(\varpi_n + \tau_0/n)$, which is bounded in probability because $Q_n/\sqrt{n}$ and $\sqrt{n}\,\varpi_n$ are.

(ii) Each investor's $\bar{\varpi}_n^\circ(\hat{c}_i)$ is subtracted once from his own transfer and added, with weight $1/(n-1)$, to each of the $n-1$ others', so the transfers sum to zero. When participation binds and the others report truthfully, an investor who reports a value $\hat{w}$ of a place in A over one in B leaves the others a total cost of $C_{-i}^B + \varpi_i^\circ \mathbf{1}\{\varpi_i^\circ < \hat{w}\}$, where C_{-i}^B is their least cost when he is placed in B. Because ϖ_i° is symmetric about zero, $\mathbb{E}[\varpi_i^\circ \mathbf{1}\{\varpi_i^\circ < \hat{w}\}] = -\bar{\varpi}_n^\circ(|\hat{w}|)$ whichever project he reports, so that the amount $\bar{\varpi}_n^\circ(\hat{c}_i)$ he receives is, up to a constant, the expected value to the others of the assignment his report induces, and his expected payoff is, up to a constant, the expected value to everyone of that assignment, which a truthful report maximizes (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979). A fee that depends on no report leaves this argument unchanged.

Under a retained veto with truthful reports, an investor placed in his favored project bears $\langle \bar{\varpi}^\circ \rangle_{-i} - \bar{\varpi}_n^\circ(c_i)$, where $\langle \bar{\varpi}^\circ \rangle_{-i}$ is the average of $\bar{\varpi}_n^\circ(c_k)$ over the others, and an investor who is moved bears $c_i + \langle \bar{\varpi}^\circ \rangle_{-i} - \bar{\varpi}_n^\circ(c_i)$. Let $\langle \bar{\varpi}^\circ \rangle = \frac{1}{n}\sum_k \bar{\varpi}_n^\circ(c_k)$, so that $|\langle \bar{\varpi}^\circ \rangle_{-i} - \langle \bar{\varpi}^\circ \rangle| \le \bar{\varpi}_n^\circ(0)/(n-1)$, and write ϕ_Z for the standard normal density and $H(s) = (s - \phi_Z(s))^+$.

First, for $s \ge 0$, $\sqrt{n}\, g_c(0)\, \bar{\varpi}_n^\circ(s/(\sqrt{n}\, g_c(0))) = \mathbb{E}[Z_n \mathbf{1}\{Z_n > s\}]$, which converges to $\mathbb{E}[Z\mathbf{1}\{Z > s\}] = \phi_Z(s)$ by the uniform integrability of $Z_n \mathbf{1}_{\mathcal{H}_n}$ and the bound on $\mathbb{E}[|\varpi_i^\circ| \mathbf{1}_{\mathcal{H}_n^c}]$. These functions are nonincreasing in s, the limit is continuous, and $\mathbb{E}[Z_n \mathbf{1}\{Z_n > s\}\mathbf{1}_{\mathcal{H}_n}] \le \mathbb{E}[Z_n^2 \mathbf{1}_{\mathcal{H}_n}]/s$, so the convergence is uniform in $s \ge 0$. In particular $\sqrt{n}\, \bar{\varpi}_n^\circ(0) \to 1/(\sqrt{2\pi}\, g_c(0))$. With c_1 drawn from G_c independently of ϖ_i°, $\mathbb{E}[\bar{\varpi}_n^\circ(c_1)] = \mathbb{E}[\varpi_i^{\circ +} G_c(\varpi_i^{\circ +})]$, and $nG_c(\varpi_i^{\circ +})\varpi_i^{\circ +}$ converges in distribution to $(Z^+)^2/g_c(0)$ and, on $\mathcal{H}_n$, is at most $2g_c(0)\, n\varpi_i^{\circ 2}$, so that $n\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)] \to \mathbb{E}[(Z^+)^2]/g_c(0) = 1/(2g_c(0))$; and $\mathrm{Var}(n\langle \bar{\varpi}^\circ \rangle) \le n\, \bar{\varpi}_n^\circ(0)\, \mathbb{E}[\bar{\varpi}_n^\circ(c_1)] \to 0$. Hence $n\langle \bar{\varpi}^\circ \rangle \to 1/(2g_c(0))$ in probability.

Second, consider the investors placed in their favored projects. The sum of the positive parts of their burdens differs from $\sum_i (\langle \bar{\varpi}^\circ \rangle - \bar{\varpi}_n^\circ(c_i))^+$, taken over all investors, by at most $n\bar{\varpi}_n^\circ(0)/(n-1) + D_n \langle \bar{\varpi}^\circ \rangle$, which tends to zero in probability. Moreover $\sum_i (\langle \bar{\varpi}^\circ \rangle - \bar{\varpi}_n^\circ(c_i))^+ = n\langle \bar{\varpi}^\circ \rangle - \sum_i \min\{\langle \bar{\varpi}^\circ \rangle, \bar{\varpi}_n^\circ(c_i)\}$, and

$$\sum_i \min\{\langle \bar{\varpi}^\circ \rangle, \bar{\varpi}_n^\circ(c_i)\} \;\le\; \langle \bar{\varpi}^\circ \rangle \#\{i : c_i \le n^{-1/4}\} + \sum_i \bar{\varpi}_n^\circ(c_i) \mathbf{1}\{c_i > n^{-1/4}\}.$$

The first term is of order $n^{-1/4}$ in probability. The second has expectation at most $n\, \bar{\varpi}_n^\circ(n^{-1/4})$, which is at most $\sqrt{n}\,\mathbb{E}[Z_n^4 \mathbf{1}_{\mathcal{H}_n}]/(g_c(0)^4 n^{3/4}) + n\,\mathbb{E}[|\varpi_i^\circ| \mathbf{1}_{\mathcal{H}_n^c}]$ and tends to zero. Hence the positive parts of these burdens sum to $1/(2g_c(0)) + o_p(1)$.

Third, consider the investors moved. Let the over-subscribed project have N supporters with costs $X_{(1)} \le \cdots \le X_{(N)}$, of whom the $D = D_n$ cheapest are moved, and let $s_l = \sqrt{n}\, g_c(0) X_{(l)}$. The positive parts of their burdens sum to $\sum_{l \le D} (X_{(l)} - \bar{\varpi}_n^\circ(X_{(l)}))^+$ up to $D(\langle \bar{\varpi}^\circ \rangle + \bar{\varpi}_n^\circ(0)/(n-1))$, which tends to zero in probability, and by the first step $\sqrt{n}\, g_c(0)$ times that sum is $\sum_{l \le D} H(s_l) + D\, o(1)$, with $D = O_p(\sqrt{n})$. Let $F_n(s) = n^{-1/2} \#\{l \le D : s_l \le s\}$. Given the labels, the number of supporters with $s_l \le s$ is binomial with N trials and probability $G_c(s/(\sqrt{n}\, g_c(0)))$, so that, divided by $\sqrt{n}$, it converges in probability to $s/2$, and $F_n(s)$, the smaller of that number and D divided by $\sqrt{n}$, converges to $\min\{|Z|, s\}/2$ jointly with $D_n/\sqrt{n} \Rightarrow |Z|/2$; because F_n is nondecreasing and the limit

is continuous, the convergence is uniform on compact sets. Since H is Lipschitz on compact sets and $s_D = \sqrt{n}\, g_c(0)\varpi_n$ is bounded in probability, integration by parts on $[0, S]$, with S large, shows that $n^{-1/2} \sum_{l \le D} H(s_l) = \int H \, dF_n$ converges in distribution to $\frac{1}{2} \int_0^{|Z|} H(s)\, ds$. Hence the positive parts of the moved investors' burdens sum to $\frac{1}{2g_c(0)} \int_0^{|Z|} H(s)\, ds + o_p(1)$, jointly with the limits of Theorem 19(v).

Finally, write b_i for investor i's burden. Every burden is at most $\varpi_n + 2\bar{\varpi}_n^\circ(0)$ in absolute value, which tends to zero in probability, and $G_v(x) = g_v(0)x(1 + o(1))$ as x falls to zero, so that $\mu_n = \sum_i G_v(b_i^+)$ converges in distribution to $\mu = \frac{\gamma}{2}(1 + \int_0^{|Z|} H(s)\, ds)$, jointly with the limits of Theorem 19(v). Given costs and favored projects, Q_n is a sum of independent Bernoulli variables with probabilities $G_v(b_i^+)$, and by the inequality of Le Cam (1960) its conditional distribution differs in total variation from the Poisson distribution with mean μ_n by at most $\sum_i G_v(b_i^+)^2 \le \mu_n \max_i G_v(b_i^+)$, which tends to zero in probability. For bounded continuous k_1 and k_2, and V_n the vector of the variables of Theorem 19(v), which depend only on costs and favored projects, $\mathbb{E}[k_1(Q_n)k_2(V_n)]$ therefore differs by $o(1)$ from $\mathbb{E}[\pi_{k_1}(\mu_n)k_2(V_n)]$, where $\pi_{k_1}(m)$ is the expectation of k_1 under the Poisson distribution with mean m, a bounded continuous function of m; hence (Q_n, V_n) converges in distribution to a Poisson variable with the random mean μ, jointly with the limit of V_n. Since $s - \phi_Z(s)$ is increasing and vanishes at $s_0 = 0.372$, $\int_0^z H(s)\, ds = (z^2 - s_0^2)/2 - \Phi_Z(z) + \Phi_Z(s_0)$ for $z \ge s_0$ and zero for $z < s_0$, where Φ_Z is the standard normal distribution function, and numerical integration over $|Z|$ gives $\mathbb{E}[\mu] = 0.659\gamma$.

For the fee, τ_0/n raises every burden by τ_0/n, so that the second step gives $1/(2g_c(0)) + \tau_0$ in place of $1/(2g_c(0))$, the third is unchanged, and the limit holds with $g_v(0)\tau_0$ added to its mean. Every transfer is at most $\bar{\varpi}_n^\circ(0) + \tau_0/n$ in absolute value. Given costs and favored projects, the expected number of withdrawals is $\sum_i G_v(b_i^+)$, where $b_i^+ \le \langle \bar{\varpi}^\circ \rangle_{-i} + \tau_0/n$ for an investor who is not moved and $b_i^+ \le \varpi_n + \bar{\varpi}_n^\circ(0) + \tau_0/n$ for one who is; for large n, therefore, $\mathbb{E}[Q_n] \le 2g_v(0)\big(n\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)] + \tau_0 + \mathbb{E}[D_n(\varpi_n + \bar{\varpi}_n^\circ(0) + \tau_0/n)]\big) + n\mathbb{P}(\varpi_n > x_1/2)$, which is bounded, because $n\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)]$ converges, $\sqrt{n}\,\bar{\varpi}_n^\circ(0)$ and $\mathbb{E}[D_n]/\sqrt{n}$ converge, and $\mathbb{E}[D_n\varpi_n]$ is bounded: on the event that $\varpi_n \le x_1/2$ by the Cauchy–Schwarz inequality, since $\mathbb{E}[D_n^2] = O(n)$ and Υ is finite, and off it because $D_n\varpi_n \le n\varpi_n$ and $n\,\mathbb{E}[\varpi_n \mathbf{1}\{\varpi_n > x_1/2\}] \to 0$. The transfers of the investors who withdraw therefore have expected absolute sum at most $(\bar{\varpi}_n^\circ(0) + \tau_0/n)\,\mathbb{E}[Q_n] = O(n^{-1/2})$, while the transfers of all placed investors sum to τ_0, so that the expected transfers collected from those who accept tend to $\tau_0 > 0$.

(iii) Let the others report truthfully, and let investor i favor A with cost $c' \ge c$ and margin v. Reporting truthfully, he bears at least $-\bar{\varpi}_n^\circ(c') \ge -\bar{\varpi}_n^\circ(c)$ whether or not he is moved, because $\langle \bar{\varpi}^\circ \rangle_{-i} \ge 0$, his cost is nonnegative and $\bar{\varpi}_n^\circ$ is nonincreasing, so that his expected payoff is at most $v + \bar{\varpi}_n^\circ(c)$. Reporting that he favors A at a cost of zero, he is placed in A exactly when $\varpi_i^\circ < 0$, an event of probability $\frac{1}{2}$ because M is symmetric about $(n-1)/2$, which is not an integer, and he then bears $\langle \bar{\varpi}^\circ \rangle_{-i} - \bar{\varpi}_n^\circ(0)$; when he is moved he can withdraw. His expected payoff is therefore at least $\mathbb{E}[(v + \bar{\varpi}_n^\circ(0) - \langle \bar{\varpi}^\circ \rangle_{-i})\mathbf{1}\{\varpi_i^\circ < 0\}] \ge \frac{1}{2}(v + \bar{\varpi}_n^\circ(0)) - \mathbb{E}[\bar{\varpi}_n^\circ(c_1)]$, and the report gains whenever $v < \underline{v}_n = \bar{\varpi}_n^\circ(0) - 2\bar{\varpi}_n^\circ(c) - 2\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)]$. By the first step of (ii), $\sqrt{n}\,\bar{\varpi}_n^\circ(0) \to 1/(\sqrt{2\pi}\, g_c(0))$, $\sqrt{n}\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)] \to 0$, and $\sqrt{n}\, g_c(0)\bar{\varpi}_n^\circ(c) = \mathbb{E}[Z_n \mathbf{1}\{Z_n > \sqrt{n}\, g_c(0)c\}] \to 0$, since the convergence

to ϕ_Z is uniform and $\phi_Z(s) \to 0$ as $s \to \infty$; hence $\sqrt{n}\,\underline{v}_n \to 1/(\sqrt{2\pi}\,g_c(0))$, and $\underline{v}_n > 0$ for all large n. The case of an investor who favors B is symmetric. With the fee of part (ii), the truthful payoff is at most $(v + \bar{\varpi}_n^\circ(c) - \tau_0/n)_+$ and the deviation's at least $\frac{1}{2}(v + \bar{\varpi}_n^\circ(0) - \tau_0/n) - \mathbb{E}[\bar{\varpi}_n^\circ(c_1)]$, since the deviator who withdraws pays no fee; when the first is zero the second is positive for all large n and every $v \geq 0$, because $\sqrt{n}\,\bar{\varpi}_n^\circ(0)$ converges to a positive limit while $\sqrt{n}\,\mathbb{E}[\bar{\varpi}_n^\circ(c_1)]$ and $\sqrt{n}\,\tau_0/n$ tend to zero, and otherwise the gain exceeds the gain without the fee by $\tau_0/(2n)$, so that the argument goes through with $\underline{v}_n$ raised by τ_0/n, which has the same limit. The expected number of investors whose costs are at least c and whose margins are below $\underline{v}_n$ is $n(1 - G_c(c))G_v(\underline{v}_n)$, and $nG_v(\underline{v}_n)/\sqrt{n} \to g_v(0)/(\sqrt{2\pi}\,g_c(0)) = \gamma/\sqrt{2\pi}$. □

Proposition 26

Reports are truthful. By the proof of Proposition 25(i), for each investor i there is a number $\underline{\tau}_i$, determined by the others' reports, such that i's transfer is $\underline{\tau}_i$ when he is placed in the project that is not crowded without him and $\underline{\tau}_i + |\varpi_i^\circ|$ when he is placed in the one that is. Consider a state in which A is crowded by $D \geq 2$, the case of B being symmetric and the states with $D \leq 1$ having probability tending to zero. Then every investor faces a positive price: an investor *kept* in A, who favors A with a cost above $X_{(D)}$, faces $X_{(D)} = \varpi_n$ and bears $\underline{\tau}_i + \varpi_n$; an investor on the *uncrowded side*, who favors B, faces $X_{(D+1)}$ and bears $\underline{\tau}_i$; and an investor *moved*, who favors A with a cost below $X_{(D)}$, faces $X_{(D+1)}$ and bears $c_i + \underline{\tau}_i$. Call a kept investor *wrong* if his burden is at least $\varpi_n/4$ and an investor on the uncrowded side wrong if his burden is at most $-\varpi_n/4$, let n_{w} be the number of wrong investors, and for $\alpha \in (0, \frac{1}{4})$ let $\mathcal{P}_\alpha$ be the set of states in which the crowded project is over-subscribed by at least two and $n_{\mathrm{w}} < \alpha n$.

The states in which the mechanism is mostly right have probability at most one half. For a state r and an investor i, let $r^{(i)}$ be the state in which i's report is replaced by an independent copy. Suppose r and $r^{(i)}$ both lie in $\mathcal{P}_\alpha$, with i kept in one of them and on the uncrowded side in the other; the same project is crowded in both, since changing one report changes the over-subscription by at most one. The number $\underline{\tau}_i$ is the same in both. If i is kept in r and not wrong there, then $\underline{\tau}_i < -3\varpi_n(r)/4$; in $r^{(i)}$ he favors B, the supporters of A are those of r less an investor whose cost exceeded $X_{(D)}$, so that the price is $X_{(D-1)}(r) \leq \varpi_n(r)$, and if he is not wrong there either, $\underline{\tau}_i > -X_{(D-1)}(r)/4 \geq -\varpi_n(r)/4$, which is impossible. If instead i is on the uncrowded side in r and kept in $r^{(i)}$, then not being wrong in r gives $\underline{\tau}_i > -\varpi_n(r)/4$, while in $r^{(i)}$ the supporters of A are those of r and one more, so that the price is at least $\varpi_n(r)$ and not being wrong gives $\underline{\tau}_i < -3\varpi_n(r)/4$, again impossible. Hence i is wrong in r or in $r^{(i)}$, and writing $\mathfrak{p}_i$ for the probability that r and $r^{(i)}$ both lie in $\mathcal{P}_\alpha$ with i kept in one and on the uncrowded side in the other,

$$\sum_i \mathfrak{p}_i \leq \sum_i \Big(\mathbb{E}[\mathbf{1}\{i \text{ wrong in } r\}\mathbf{1}_{\mathcal{P}_\alpha}(r)] + \mathbb{E}[\mathbf{1}\{i \text{ wrong in } r^{(i)}\}\mathbf{1}_{\mathcal{P}_\alpha}(r^{(i)})]\Big) = 2\,\mathbb{E}[n_{\mathrm{w}}\mathbf{1}_{\mathcal{P}_\alpha}] \leq 2\alpha n\,\mathbb{P}(\mathcal{P}_\alpha),$$

because $r^{(i)}$ has the law of r. Given the others' reports, i's two reports are independent, so $\mathfrak{p}_i = 2\,\mathbb{E}[a_i^{\mathrm{k}} a_i^{\mathrm{u}}]$, where a_i^{k} and a_i^{u} are the conditional probabilities, given the others' reports, that

$r \in \mathcal{P}_\alpha$ with i kept and that $r \in \mathcal{P}_\alpha$ with i on the uncrowded side. Each is at most $\frac{1}{2}$, the conditional probability that i favors the relevant project, and their sum is at least $q_i - \frac{1}{2}G_c(|\varpi_i^\circ|)$, where q_i is the conditional probability of $\mathcal{P}_\alpha$ and $\frac{1}{2}G_c(|\varpi_i^\circ|)$ that of i being moved. For $a, b \in [0, \frac{1}{2}]$, $ab \geq \frac{1}{2}(a + b - \frac{1}{2})^+$, so that, by Jensen's inequality, $\mathbb{E}[a_i^{\mathrm{k}} a_i^{\mathrm{u}}] \geq \frac{1}{2}(\mathbb{P}(\mathcal{P}_\alpha) - \frac{1}{2} - \frac{1}{2}\mathbb{E}[G_c(|\varpi_1^\circ|)])^+$. Summing over i and comparing with the bound above,

$$\mathbb{P}(\mathcal{P}_\alpha) \;\leq\; \frac{\frac{1}{2} + \frac{1}{2}\mathbb{E}[G_c(|\varpi_1^\circ|)]}{1 - 2\alpha},$$

and $\mathbb{E}[G_c(|\varpi_1^\circ|)] \to 0$ because $\varpi_1^\circ \to 0$ in probability.

A mostly wrong state draws on capacity. Let $\lambda < \lambda_v$ and let $\mathcal{L}$ and $\underline{\mathcal{L}}$ be as in the proof of Proposition 25(i). In a state r, write $\mathbb{E}[\,\cdot \mid r]$ for the expectation over margins, and b_i and τ_i for the burdens and transfers, which r determines. Then $\mathbb{E}[Q_n \mid r] - \lambda\,\mathbb{E}[T_n \mid r] = \sum_i \mathcal{L}(b_i, \tau_i) \geq \sum_i \underline{\mathcal{L}}(b_i)$, every term of which is nonnegative. A wrong kept investor has $b_i \geq \varpi_n/4$ and contributes at least $(1 - \lambda/\kappa_v)G_v(\varpi_n/4)$, and a wrong investor on the uncrowded side has $b_i \leq -\varpi_n/4$ and contributes at least $\lambda\varpi_n/4$. Hence, in a state with $n_{\mathrm{w}} \geq \alpha n$ and $\varpi_n \leq x_1$,

$$\mathbb{E}[Q_n \mid r] + \lambda\,\mathbb{E}[\Delta_n \mid r] \;\geq\; \mathbb{E}[Q_n \mid r] - \lambda\,\mathbb{E}[T_n \mid r] \;\geq\; \alpha n\, \mathfrak{c}\, \varpi_n, \qquad \mathfrak{c} = \tfrac{1}{4}\min\{(1 - \lambda/\kappa_v)g_v(0)/2,\ \lambda\},$$

since $G_v(x) \geq g_v(0)x/2$ on $[0, x_1]$, with x_1 as in the proof of Proposition 25.

Given the state, the capacity concentrates. Given r, Q_n is a sum of independent indicators with mean $m = \sum_i G_v(b_i^+)$, so that $\mathbb{P}(Q_n \leq m/2 \mid r) \leq 4/m$, and T_n is a sum of independent terms $\tau_i \mathbf{1}\{i \text{ accepts}\}$ with variance $\sum_i \tau_i^2 G_v(b_i^+)(1 - G_v(b_i^+))$. A term with $b_i \leq 0$ vanishes. If $b_i > 0$ and $\tau_i \geq 0$, then $\tau_i \leq b_i$ and $b_i^2(1 - G_v(b_i)) \leq \mathbb{E}[v^2]$ by Markov's inequality, so the term is at most $\mathbb{E}[v^2]G_v(b_i)$; if $b_i > 0 > \tau_i$, then i is moved, since $b_i - \tau_i$ is the cost of his place, so that $|\tau_i| < c_i \leq \varpi_n$ and the term is at most $\varpi_n^2 G_v(b_i)$. Hence $\mathrm{Var}(T_n \mid r) \leq (\mathbb{E}[v^2] + \varpi_n^2)\, m$.

(i) Fix $\delta < \frac{1}{2}$ and choose $\alpha \in (0, \frac{1}{4})$ and $\eta > 0$ with $(\frac{1}{2} - 2\alpha)/(1 - 2\alpha) - 2\eta g_c(0)/\sqrt{2\pi} > \delta$. Let $\mathcal{A}_n$ be the event that the crowded project is over-subscribed by at least two, $n_{\mathrm{w}} \geq \alpha n$, $\eta \leq \sqrt{n}\,\varpi_n$ and $\varpi_n \leq x_1$. Its complement is contained in the union of $\mathcal{P}_\alpha$, the event $\sqrt{n}\,\varpi_n < \eta$, whose probability tends to $\mathbb{P}(|Z| < \eta g_c(0)) \leq 2\eta g_c(0)/\sqrt{2\pi}$, and two events of vanishing probability, so that $\liminf \mathbb{P}(\mathcal{A}_n) \geq (\frac{1}{2} - 2\alpha)/(1 - 2\alpha) - 2\eta g_c(0)/\sqrt{2\pi} > \delta$. On $\mathcal{A}_n$, with $\mathfrak{b} = \alpha\mathfrak{c}\eta/2$, which depends on the mechanism through nothing, the second step gives $\mathbb{E}[Q_n \mid r] + \lambda\,\mathbb{E}[\Delta_n \mid r] \geq 2\mathfrak{b}\sqrt{n}$. If $m \geq \mathfrak{b}\sqrt{n}$, then $\mathbb{P}(Q_n \geq \mathfrak{b}\sqrt{n}/2 \mid r) \geq 1 - 4/(\mathfrak{b}\sqrt{n})$. If $m < \mathfrak{b}\sqrt{n}$, then $-\mathbb{E}[T_n \mid r] \geq \mathfrak{b}\sqrt{n}/\lambda$, and since $\Delta_n \geq -T_n$, Chebyshev's inequality and the third step give $\mathbb{P}(\Delta_n < \mathfrak{b}\sqrt{n}/(2\lambda) \mid r) \leq \mathbb{P}(T_n \geq \mathbb{E}[T_n \mid r] + \mathfrak{b}\sqrt{n}/(2\lambda) \mid r) \leq 4\lambda^2(\mathbb{E}[v^2] + x_1^2)/(\mathfrak{b}\sqrt{n})$. In either case $\mathbb{P}(Q_n + \lambda\Delta_n \geq \mathfrak{b}\sqrt{n}/2 \mid r) \geq 1 - C/\sqrt{n}$ on $\mathcal{A}_n$, so that $\liminf \mathbb{P}(Q_n + \lambda\Delta_n \geq \mathfrak{b}\sqrt{n}/2) \geq \liminf \mathbb{P}(\mathcal{A}_n) > \delta$, and for all large n the $(1 - \delta)$-quantile of $Q_n + \lambda\Delta_n$ is at least $\mathfrak{b}\sqrt{n}/2$, uniformly over the mechanisms of the class.

(ii) The transfer is $\tau_i = (\varpi_i^\circ)^+\mathbf{1}\{i \text{ in } A\} + (\varpi_i^\circ)^-\mathbf{1}\{i \text{ in } B\} - (\varpi_i^\circ)^+\mathbf{1}\{N_{-i}^A \text{ even}\} - (\varpi_i^\circ)^-\mathbf{1}\{N_{-i}^B \text{ even}\}$, where N_{-i}^A and N_{-i}^B count the other investors favoring A and B. The last two terms do not depend on i's report, and the first two differ, between A and B, by ϖ_i°, so that truthful reporting is dominant in the assignment by the argument of the proof of Proposition 20. Let A be crowded

by $D \geq 1$, so that every investor faces a positive price, and let n_A be odd. An investor who favors A has $N_{-i}^A = n_A - 1$ even and transfer $\varpi_i^\circ \mathbf{1}\{i \text{ in } A\} - \varpi_i^\circ$: kept, he bears nothing; moved, he receives $X_{(D+1)}$ and bears $c_i - X_{(D+1)} \leq 0$. An investor who favors B has $N_{-i}^A = n_A$ odd and transfer zero. Since margins are positive, nobody withdraws, $Q_n = 0$, and $\Delta_n = D_n X_{(D+1)}$, the price paid to the investors moved; the case of B crowded with n_B odd is symmetric, and n_A is odd exactly when n_B is, because n is even. Now $D_n X_{(D+1)} = D_n \varpi_n + D_n(X_{(D+1)} - X_{(D)})$, where $D_n \varpi_n \Rightarrow Z^2/(2g_c(0))$ by Theorem 19(v) and the spacing is of order $1/n$ in probability, as in the proof of Proposition 24(ii), so that $D_n X_{(D+1)} \Rightarrow Z^2/(2g_c(0))$. The parity of n_A is a function of D_n and of the crowded project, so its asymptotic independence from $D_n X_{(D+1)}$ needs two facts: $\mathbb{P}(n_A = k+1)/\mathbb{P}(n_A = k) = (n-k)/(k+1)$ tends to one uniformly over $|k - n/2| \leq C\sqrt{n}$, and, conditionally on the labels, $h_n(d) = \mathbb{P}(dX_{(d+1)} \leq s \mid D_n = d)$ varies slowly in d, because under the representation of the proof of Theorem 19(v) the difference between $(d+1)X_{(d+2)}$ and $dX_{(d+1)}$ is of order d/N in probability, uniformly over $d \leq C\sqrt{n}$, and the limit law of $D_n X_{(D+1)}$ is continuous; pairing consecutive values of d then gives $\mathbb{P}(n_A \text{ odd}, D_n X_{(D+1)} \leq s) \to \frac{1}{2}\mathbb{P}(Z^2 \leq 2g_c(0)s)$ for every $s > 0$. In a state with n_A even and $D_n \geq 1$ the K kept investors each withdraw with probability $G_v(\varpi_n)$, so $\mathbb{P}(Q_n = 0 \mid r) \leq e^{-KG_v(\varpi_n)}$, which tends to zero in probability because $\sqrt{n}\,\varpi_n$ is bounded away from zero in probability, and the states with $D_n = 0$ have probability of order $n^{-1/2}$; hence $\mathbb{P}(Q_n = 0, \Delta_n \leq s)$ has the same limit, and for $\delta > \frac{1}{2}$ a bounded s with $\frac{1}{2}\mathbb{P}(Z^2 \leq 2g_c(0)s) > 1 - \delta$ gives $\mathcal{S}_n(\delta) \leq \lambda s$ for all large n. When n_A is even, an investor who favors A has transfer $\varpi_i^\circ \mathbf{1}\{i \text{ in } A\}$, so that a kept investor bears ϖ_n and a moved one his cost, and an investor who favors B receives $X_{(D+1)}$; hence $\mathbb{E}[Q_n \mid r] = KG_v(\varpi_n) + \sum_{\text{moved}} G_v(c_i)$, which is $\frac{n}{2}g_v(0)\varpi_n(1 + o_p(1))$, while in a state with $D_n = 0$ nobody pays and $Q_n = 0$; the same asymptotic independence of the parity, together with the uniform integrability of $\sqrt{n}\,\varpi_n \mathbf{1}_{\mathcal{E}_n}$ from the proof of Theorem 19(v), gives $\mathbb{E}[Q_n]/\sqrt{n} \to \frac{1}{2} \cdot \frac{1}{2} g_v(0)\,\mathbb{E}|Z|/g_c(0) = \gamma\,\mathbb{E}|Z|/4$. □

Corollary 27

Let A be crowded by $D \geq 1$, so that it has $K + D$ supporters, and consider one of them, with cost c_i. Given the others' reports, the price he faces, ϖ_i°, is the D-th smallest reported cost among the other $K + D - 1$ supporters. He is moved exactly when his report is below ϖ_i°, since the D lowest reports among the $K + D$ supporters are then his and the $D - 1$ below ϖ_i° among the others, and he is then paid the $(D+1)$-th smallest of all reports, which is ϖ_i°; otherwise he is kept and pays nothing. His payoff is therefore $\varpi_i^\circ - c_i$ if moved and zero if kept, and reporting c_i moves him exactly when $\varpi_i^\circ > c_i$, so truthful reporting is dominant, by the argument of Vickrey (1961); an investor who favors the uncrowded project makes no report that matters and pays nothing. Under a retained veto, a moved investor bears $c_i - X_{(D+1)} \leq 0$, since his cost is among the D smallest, and every other investor bears nothing, so with positive margins nobody withdraws, $Q_n = 0$, and the cash paid is $\Delta_n = D_n X_{(D+1)}$, which converges in distribution to $Z^2/(2g_c(0))$ as in the proof of Proposition 26(ii). When $n_A = K$ nothing is moved or paid, so $Q_n = \Delta_n = 0$. For the expectation, let $\mathcal{E}_n$ be the event that $D \leq N^{2/3}$ and $U_{(D+1)} \leq u_\epsilon$, in the representation and with the u_ϵ of the proof of Theorem 19(v);

on it $X_{(D+1)} \le 2U_{(D+1)}/g_c(0)$ and $\mathbb{E}[U^2_{(D+1)} \mid D, N] = (D+1)(D+2)/((N+1)(N+2))$, so that, since $\mathbb{E}[D_n^4] = O(n^2)$ and $N \ge n/2$, $D_n X_{(D+1)} \mathbf{1}_{\mathcal{E}_n}$ has a bounded second moment, by the same computations as in that proof and in the proof of Proposition 24(ii), and the contribution of $\mathcal{E}_n^c$ vanishes by the same tail estimates; hence $\mathbb{E}[\Delta_n] \to 1/(2g_c(0))$. With costs uniform on $[0,1]$ and $n = 2k$, given $D = d$ the expected $(d+1)$-th smallest of $k+d$ uniform variables is $(d+1)/(k+d+1)$, so $\mathbb{E}[\Delta_n] = \mathbb{E}[D(D+1)/(K+D+1)]$, which is twice the sum evaluated at the end of the proof of Theorem 19(v), that is $n/(2(n+1))$. □

Corollary 42

Under binding enrollment the n placed investors report after the closing is settled, and the others' reports have the law assumed in the proof of Proposition 25(ii), so truthful reporting is a Bayesian equilibrium among them by that proof. The transfers sum to zero over the placed investors, so their net burdens sum to the cost of the moves, C_n. The placed investors are drawn at random among those who enroll, independently of margins, so by exchangeability each placed investor's expected net burden is $\mathbb{E}[C_n]/n$ whatever the others' enrollment decisions, and the argument of the proof of Theorem 22(iii) gives the threshold rule, its optimality against every enrollment behavior of the others, and the Poisson limit with mean $g_v(0) \lim \mathbb{E}[C_n] = \gamma/4$. □

Proposition 28

Given which project is good, favored projects are independent, so the likelihood of the observed favored projects when A is good, relative to when B is, is $((\frac{1}{2}+\varepsilon)/(\frac{1}{2}-\varepsilon))^{n_A - n_B} = \ell_\varepsilon^{d}$ with $\ell_\varepsilon = (1+2\varepsilon)/(1-2\varepsilon)$, and with equal priors the posterior probability that the less favored project is the good one is $1/(1+\ell_\varepsilon^{|d|})$. A unit of it is therefore worth $(v^+ + \ell_\varepsilon^{|d|} v^-)/(1+\ell_\varepsilon^{|d|})$ in expectation, which is positive exactly when $\ell_\varepsilon^{|d|} < v^+/(-v^-)$, that is, when $|d| \log \ell_\varepsilon < \log(v^+/(-v^-))$. The right side is positive because $v^+ + v^- > 0$, and $\log \ell_\varepsilon$ rises with ε from zero, which gives the comparative statics. Given the good project, d/n converges almost surely to 2ε if it is A and to -2ε if it is B, so $\mathbb{P}(|d| < \bar{d}(\varepsilon)) \to 0$ and the less favored project is the bad one with probability tending to one. □

Proposition 43

(i) The statements on B^* are Proposition 16. Widening permissions lowers $R^*(\Gamma, \mathcal{W})$ by Proposition 17(ii), and adding a willing investor enlarges $\mathcal{W}$ and so, by part (i) of that proposition, weakly lowers it. For each S, $\sum_{i \in S} h_i - \max_{i \in S} h_i$ is nondecreasing in each h_i, because raising h_i by ϵ raises the sum by ϵ and the maximum by at most ϵ, so the minimum over S is nondecreasing as well. (ii) For $\eta > 0$ the limit in Theorem 18(iii) is positive, and its derivative in η is $\rho_r(1-\rho_r)/(\eta + (1-\eta)\rho_r)^2 > 0$. When the recommendation shifts favored projects in the same way, the share vector $(1-\eta)\rho + \eta e_r$, with e_r placing all weight on r, differs from ρ, and Theorem 19(iv) gives a positive limit per investor. (iii) If $\mathcal{W}' \subseteq \mathcal{W}$, then $\mathcal{F}(\Gamma, \mathcal{W}') \subseteq \mathcal{F}(\Gamma, \mathcal{W})$ and the minimum

defining R^* is taken over fewer sets; removing investors removes terms from the sums $\sum_{i\in\mathcal{N}(\mathcal{J}')} q_i$, so B^* weakly rises. □

Theorem 29

Let Π_0 be the finite set of pure plans available on $\mathcal{M}$, let $\mathcal{R}_i$ be the finite set of account i's records, and let $\Pi_0^d \subseteq \Pi_0$ be the pure plans that do not replicate the target. Under own-account compensation $\phi = (\phi_i)_i$, $\phi_i \in \mathbb{R}^{\mathcal{R}_i}$, a plan π pays the manager $\sum_i \langle \phi_i, \mu_i^\pi \rangle - c_\pi$, which is linear in the weights of a mixed plan, so the target is optimal if and only if

$$\sum_i \langle \phi_i, \mu_i^\tau - \mu_i^\pi \rangle \;\geq\; c_{e^*} - c_\pi \qquad \text{for every } \pi \in \Pi_0, \tag{14}$$

and every optimal plan replicates the target if and only if, in addition, the inequality is strict for every $\pi \in \Pi_0^d$, because a mixed plan is optimal only if every pure plan it uses is.

(i) By Farkas's lemma, (14) has no solution if and only if there are weights $\lambda_\pi \geq 0$ with $\sum_\pi \lambda_\pi(\mu_i^\tau - \mu_i^\pi) = 0$ for every i and $\sum_\pi \lambda_\pi(c_{e^*} - c_\pi) > 0$. The second condition makes $s = \sum_\pi \lambda_\pi$ positive, and the plan that mixes the pure plans with weights λ_π / s replicates the target and costs less. Conversely, every own-account compensation pays a replicating plan exactly what it pays the target, so a cheaper one defeats the target.

(ii) Consider the linear program that maximizes $\varepsilon \leq 1$ subject to $\sum_i \langle \phi_i, \mu_i^\tau - \mu_i^\pi \rangle - \varepsilon\, \mathbf{1}\{\pi \in \Pi_0^d\} \geq c_{e^*} - c_\pi$ for every $\pi \in \Pi_0$. Strict implementation is possible if and only if its value is positive. Its dual minimizes $z - \sum_\pi \lambda_\pi(c_{e^*} - c_\pi)$ over $\lambda \geq 0$ and $z \geq 0$ subject to $\sum_\pi \lambda_\pi(\mu_i^\tau - \mu_i^\pi) = 0$ for every i and $\sum_{\pi\in\Pi_0^d} \lambda_\pi + z = 1$, so the value is not positive if and only if such weights satisfy $z \leq \sum_\pi \lambda_\pi(c_{e^*} - c_\pi)$. If $\sum_{\Pi_0^d} \lambda_\pi = 0$, then $z = 1$ and the weights form a cheaper replicating plan, which (i) excludes; otherwise they form a replicating plan that costs no more than the target and places positive weight on Π_0^d. The converse is immediate, since such a plan is paid exactly what the target is paid and costs no more.

If compensation must be nonnegative, Farkas's lemma for nonnegative variables replaces each equality $\sum_\pi \lambda_\pi(\mu_i^\tau - \mu_i^\pi) = 0$ by the inequality ≤ 0 pointwise; then $\sum_\pi \lambda_\pi \mu_i^\pi \geq s\mu_i^\tau$ pointwise, both sides have total mass s, so equality holds, and the same conclusions follow. □

Corollary 30

Plans available on a menu are available on any larger menu, so the conditions of Theorem 29(ii) on X imply them on every menu containing the target's allocations, and some own-account compensation strictly implements the target there. Every optimal plan then replicates the target, so it gives each investor the same law of his record, hence, with complete records, the same law of terminal wealth before payments, and under own-account compensation his payment is a function of his record, so the joint law of his wealth and payment is also the same. □

Proposition 46

(i) With jointly normal returns and background wealth, investor i's certainty equivalent from exposure x_i is $x_i^\top \mu - \frac{\alpha_i}{2}(x_i^\top \Sigma x_i + 2x_i^\top \sigma_i)$ plus terms that do not depend on x_i; it is maximized at x_i^* and falls short of its maximum by $\frac{\alpha_i}{2}(x_i - x_i^*)^\top \Sigma (x_i - x_i^*) = \frac{1}{2}\| \sqrt{\alpha_i}\, \Sigma^{1/2}(x_i - x_i^*)\|^2$. With k profiles, $x_i = P^\top a_i$, and as a_i ranges over $\mathbb{R}^k$ the vector $\sqrt{\alpha_i}\, \Sigma^{1/2} x_i$ ranges over the subspace $V = \Sigma^{1/2} P^\top \mathbb{R}^k$, of dimension at most k. Writing y_i for the i-th column of the matrix whose singular values are the ν_j, the aggregate loss is at least $\frac{1}{2}\sum_i \|y_i - \Pi_V y_i\|^2$, where Π_V is the orthogonal projection onto V, with equality when each a_i attains the projection. By the theorem of Eckart and Young (1936), the least value of $\sum_i \|y_i - \Pi_V y_i\|^2$ over subspaces of dimension at most k is $\sum_{j>k} \nu_j^2$, attained by the span of the first k left singular vectors $u_1, \dots, u_k$, which the profiles $P^\top = \Sigma^{-1/2}[u_1, \dots, u_k]$ produce. Requiring long combinations shrinks the attainable exposures and cannot lower the loss.

(ii) Aggregate certainty equivalents net of the cost of records are, up to a constant, $-\frac{1}{2}\sum_{j>k} \nu_j^2 - \kappa k(n + Tm)$, whose increment from $k-1$ to k profiles is $\frac{1}{2}\nu_k^2 - \kappa(n + Tm)$. The increments are nonincreasing in k and, with $\kappa > 0$, eventually negative, so the largest maximizer over $k \geq 1$ is the largest k with a nonnegative increment, or $k = 1$ if there is none, which is k^*. A fall in κ raises every increment and a rise in T lowers every increment, which gives the comparative statics.

(iii) If $\sigma_i = \upsilon_i \sigma$ for scalars υ_i and a common vector σ, every $x_i^* = \alpha_i^{-1}\Sigma^{-1}\mu - \upsilon_i \Sigma^{-1}\sigma$ lies in the span of $\Sigma^{-1}\mu$ and $\Sigma^{-1}\sigma$, so the matrix has rank at most two, $\nu_3 = 0$, and by (i) two profiles lose nothing. In general the column y_i is $\sqrt{\alpha_i}\, \Sigma^{-1/2}(\alpha_i^{-1}\mu - \sigma_i)$, so the rank is at most one more than the dimension of the span of the background covariances, and it equals one more when μ lies outside that span and the vectors $(\alpha_i^{-1}, \sigma_i)$ span one more dimension than the σ_i, since the map $(t, s) \mapsto t\mu - s$ is then injective on their span. □

Proposition 47

Investor k accepts after the change if and only if $s_k - \Delta + \zeta_k q_k \geq 0$, and every other member still obtains $s_i + \zeta_i q_i \geq 0$ for any $\zeta_i \geq 0$, so securing k requires $\zeta_k \geq z/q_k$. Under rate matching, entitlement requires $\zeta_j \geq \zeta_k$ for every $j \in \mathcal{S}_k$; under final-terms matching it requires $\chi_j - \zeta_j \leq \chi_k - \zeta_k$, that is $\zeta_j \geq \zeta_k - \omega_j$, together with $\zeta_j \geq 0$. Write $\hat{\zeta}_j(u) = u$ under rate matching and $\hat{\zeta}_j(u) = [u - \omega_j]_+$ under final-terms matching. The cost $\sum_i \zeta_i q_i$ is then at least $\sum_{j \in \mathcal{S}_k} q_j \hat{\zeta}_j(\zeta_k)$, which is nondecreasing in ζ_k and equals $C_k(z)$ at $\zeta_k = z/q_k$, because $\omega_k = 0$. This proves necessity.

For sufficiency, set $\zeta_j = \hat{\zeta}_j(z/q_k)$ for $j \in \mathcal{S}_k$ and $\zeta_j = 0$ otherwise, at cost $C_k(z)$. Under rate matching, take any member i; every $l \in \mathcal{S}_i$ must receive at least ζ_i. If $i \in \mathcal{S}_k$, transitivity gives $\mathcal{S}_i \subseteq \mathcal{S}_k$, so every such l receives the same rate, and if $i \notin \mathcal{S}_k$, then $\zeta_i = 0$. Under final-terms matching the new charges are $\chi'_j = \min(\chi_j, \chi_k - z/q_k)$ for $j \in \mathcal{S}_k$ and $\chi'_j = \chi_j$ otherwise. Take any member i and any $l \in \mathcal{S}_i$. If $i \in \mathcal{S}_k$, then $l \in \mathcal{S}_k$ and $\chi'_l \leq \chi'_i$ because $\chi_l \leq \chi_i$; if $i \notin \mathcal{S}_k$, then $\chi'_l \leq \chi_l \leq \chi_i = \chi'_i$. Either schedule therefore honors every entitlement. The bounds $z \leq C_k(z) \leq \lambda_k z$ follow from $0 \leq [u - \omega_j]_+ \leq u$ and $\omega_k = 0$. Each term $q_j[z/q_k - \omega_j]_+$ is convex and piecewise linear with a kink at $z = q_k \omega_j$ when $\omega_j > 0$, and every term equals $q_j z/q_k$ when every ω_j is zero. □

Proposition 48

Write $\bar{D} = C_k(z) - \Phi > 0$ for the deficit. Given a waiver set J, the cheapest admissible schedule costs $C_k(z) - \sum_{j \in J} \xi_j$: a member who has waived receives no concession, every other entitled member receives what the schedule in the proof of Proposition 47 assigns him, and that schedule still honors every entitlement, because a waiver covers every entitlement that arises from the concession to k. The manager secures k if and only if this cost is at most Φ, that is, if and only if $\sum_{j \in J} \xi_j \geq \bar{D}$; because k is pivotal, the program fails otherwise and every member obtains his outside value. Members of $\mathcal{S}_k \setminus \{k\}$ with $\xi_j = 0$ are unaffected by the concession and are omitted.

(i) In a rescuing profile, a member of J whose withdrawal leaves the deficit covered gains $\xi_j + \eta_j > 0$ by withdrawing; a member whose withdrawal uncovers it would move from $s_j - \eta_j > 0$ to zero; and a member of $\mathcal{S}_k^+ \setminus J$ would lose $\xi_j + \eta_j > 0$ by waiving. A rescuing profile is therefore an equilibrium if and only if J is minimal. Because $\Phi \geq z$, $\sum_{j \in \mathcal{S}_k^+} \xi_j = C_k(z) - z \geq \bar{D}$, so $\mathcal{S}_k^+$ itself covers the deficit, and removing members one at a time while the deficit stays covered yields a minimal covering set.

(ii) At universal non-waiver the program fails. A member whose waiver would cover the deficit gains $s_j - \eta_j > 0$ by waiving; any other member would obtain $-\eta_j \leq 0$, strictly less than zero if $\eta_j > 0$. Universal non-waiver is therefore an equilibrium if and only if $\xi_j < \bar{D}$ for every $j \in \mathcal{S}_k^+$, and a strict one if every η_j is positive.

(iii) If $\xi_{j^*} \geq \bar{D}$ for some j^*, then j^* has not waived in any failing profile, and by waiving he obtains $s_{j^*} - \eta_{j^*} > 0$ instead of zero, so no failing profile is an equilibrium, while an equilibrium exists by (i). The converse is (ii).

(iv) Index the members of $\mathcal{S}_k^+$ in the order of solicitation, let A_t be the sum of ξ_l over members who waived before member t's turn, and let $B_t = \sum_{l \geq t} \xi_l$. The claim is that from any turn with $A_t + B_t \geq \bar{D}$ every subgame-perfect continuation rescues the program; it holds after the last turn, where it reduces to $A \geq \bar{D}$. Suppose it holds from turn $t+1$. If $A_t + B_{t+1} \geq \bar{D}$, member t's refusal preserves the condition, so the program is rescued whatever he does, and he strictly prefers $s_t + \xi_t$ to $s_t - \eta_t$. Otherwise only his waiver preserves the condition, and he strictly prefers the rescue it brings, worth $s_t - \eta_t > 0$, to failure. In both cases the condition holds at turn $t+1$ and the program is rescued. At the first turn $A_1 + B_1 = C_k(z) - z \geq \bar{D}$. The argument also shows that, wherever the condition holds, member t waives exactly when $A_t + B_{t+1} < \bar{D}$; where it fails the program fails whatever he does, and he does not strictly gain by waiving. □

Theorem 31, part (i)

Write $X = g(\mathcal{A}(o); o)$ and $Y = g(\mathcal{E}(o); o)$, so $Y \subseteq X$ by Lemma 2. A point of a set is undominated there if no distinct point of the set is weakly larger in every coordinate. Let $x \in X$ be undominated in X. Completion gives $y \in Y$ with $y \geq x$; since $y \in X$, $y = x$, so $x \in Y$, and x is undominated in $Y \subseteq X$. Conversely, let $y \in Y$ be undominated in Y and suppose some $x \in X$ satisfies $x \geq y$ and $x \neq y$. Completion gives $z \in Y$ with $z \geq x$, so $z \geq y$ and $z \neq y$, a contradiction. □

Corollary 33

At a formed profile, a member who declines obtains his outside value whether or not his departure causes failure, so payments to committed investors upon failure do not change his comparison, and an outsider who commits leaves the profile formed, because an additional commitment never causes a formed investment to fail, so his comparison is also unchanged; formed pure equilibria are therefore the same with and without the payments. A schedule of failure payments that implements formation has only formed equilibria, each a formed equilibrium of the baseline game, whose welfare is at most $H(o)$, and the carry on its reserve lowers welfare further. Under the hypotheses of Theorem 32, L rises by Δ and G weakly falls. □

Proposition 50

(i) Let S be an acceptable formed membership. If some outsider j obtains $u_j(S \cup \{j\}) > o_j$, admit him. By (24) every incumbent weakly gains, and the entrant strictly gains, so the enlarged membership is acceptable. The process stops after at most $n - |S|$ admissions at a membership S'. At S' no outsider strictly gains by joining. A member who leaves obtains o_i whether or not the remaining members still form the program, and $o_i \leq u_i(S')$ by acceptability, so no member strictly gains by leaving. Thus S' is a pure equilibrium weakly dominating S. An acceptable failed profile has $g_i \leq 0$ and $g_i \geq 0$ for everyone, so it gives every investor his outside value and is dominated by the all-outside equilibrium.

(ii) If S and T are willing formed memberships, $S \cup T$ is formed, and each $i \in S$ obtains $u_i(S \cup T) \geq u_i(S) \geq o_i$ by (24); the same holds for T. The union of all willing memberships, S^*, is therefore willing. No member gains by leaving. If an outsider j had $u_j(S^* \cup \{j\}) \geq o_j$, then $S^* \cup \{j\}$ would be willing, contradicting maximality, so no outsider gains by joining. Every formed equilibrium T is willing, hence $T \subseteq S^*$; members of T obtain weakly more in S^*, members of $S^* \setminus T$ obtain at least their outside values, and outsiders are unaffected. In a failed equilibrium every subscriber with $h_i > 0$ would leave, so all investors obtain their outside values, which S^* weakly exceeds.

(iii) Let S maximize $\sum_{i \in S}[u_i(S) - o_i]$ among acceptable formed memberships. If an outsider j strictly gained by joining, $S \cup \{j\}$ would be an acceptable formed membership whose surplus exceeds that of S by at least $u_j(S \cup \{j\}) - o_j > 0$. No member gains by leaving, by acceptability. Hence S is an equilibrium.

(iv) If $o' \geq o$, a membership willing at o' is willing at o, so the family of willing memberships at o' is contained in the family at o, and so is its union. □

Proposition 57

The game played by a pool P is the threshold game in which investors outside P are restricted to the null action. It satisfies Assumptions 1 and 2 and inclusion monotonicity on subsets of P, so Lemma 2, Theorem 31 and Proposition 50 apply to it, and part (i) is Lemma 2. Whether

a formed membership is willing depends only on the membership and on o, so every willing formed membership contained in P is a willing formed membership contained in P', and by Proposition 50(ii) the union of the first family is contained in the union of the second, $S^*(o;P) \subseteq S^*(o;P')$. Members of $S^*(o;P)$ weakly gain by (24), members of $S^*(o;P') \setminus S^*(o;P)$ obtain at least their outside values, which is what they obtain in the smaller pool's best equilibrium, and every other investor obtains his outside value in both. If the smaller pool contains no willing formed membership, every pure equilibrium of its game is failed and gives each investor his outside value, by the argument in the proof of Proposition 50(ii), and Lemma 2 for the larger pool gives the domination. This proves (ii). Both games have completion by Proposition 50(i), so Theorem 31 gives $H(o;Y) = W^*(o;Y)$ for $Y = P, P'$, and part (i) gives $G(o;P') - G(o;P) = H(o;P') - H(o;P)$. An acceptable profile of the smaller pool's game is acceptable in the larger pool's game, in which the added investors remain outside, so $W^*(o;P') \geq W^*(o;P)$. □

Proposition 44

For joint compensation $\Psi \geq 0$, records with $\mu^\tau(r) = 0$ add nothing to the target's payment and weakly lower the deviation's advantage, so set $\Psi = 0$ there and write $w(r) = \Psi(r)\mu^\tau(r) \geq 0$ and $\varrho(r) = \mu^\pi(r)/\mu^\tau(r)$ elsewhere. The constraint is $\sum_r w(r)[1 - \varrho(r)] \geq c$ and the expected payment is $\sum_r w(r)$; since $1 - \varrho(r) \leq 1 - \varrho$, every feasible schedule pays at least $c/(1-\varrho)$, and placing all weight on a record with $\varrho(r) = \varrho$ attains it when $\varrho < 1$. Own-account compensation is the same problem indexed by pairs (i, r_i), with weights $\phi_i(r_i)\mu_i^\tau(r_i)$ and ratios $\mu_i^\pi(r_i)/\mu_i^\tau(r_i)$. For any such pair with $\mu_i^\tau(r_i) > 0$,

$$\frac{\mu_i^\pi(r_i)}{\mu_i^\tau(r_i)} = \sum_{r \ni r_i,\ \mu^\tau(r) > 0} \frac{\mu^\tau(r)}{\mu_i^\tau(r_i)}\,\varrho(r) + \sum_{r \ni r_i,\ \mu^\tau(r) = 0} \frac{\mu^\pi(r)}{\mu_i^\tau(r_i)} \;\geq\; \varrho_{\text{joint}},$$

because the first sum is an average of joint ratios and the second is nonnegative. In the example, a concentrated account's record is right or wrong, its smallest ratio is q_0/q_1, and the joint record in which every bet is right has ratio $(q_0/q_1)^n$, the smallest; an account holding a slice of every bet records the number of right bets, and its smallest ratio is also $(q_0/q_1)^n$. □

Proposition 45

(i) If the target is optimal for the manager under $\sum_{i\in S} \phi_i$ and $S \subseteq T$, then for every plan π, $\sum_{i\in T}(\mathbb{E}_\tau\phi_i - \mathbb{E}_\pi\phi_i) \geq \sum_{i\in S}(\mathbb{E}_\tau\phi_i - \mathbb{E}_\pi\phi_i) \geq c_{e^*} - c_\pi$, because target-maximal payments make every added term nonnegative, so the target is optimal under $\sum_{i\in T}\phi_i$. A refuser's account is not managed and yields his outside value, and an engaged investor receives the value of the manager's plan net of his payments, which is the target's when the engaged accounts belong to $\mathcal{F}_\phi$.

(ii) Let all others refuse. Investor i either refuses and obtains his outside value, or engages the manager under some admissible contract. If the manager then follows the target, the contract belongs to the class over which $\underline{\phi}_i$ is defined, so its expected payment is at least $\underline{\phi}_i$, and the investor

gains at most $V_i - \underline{\phi}_i \le 0$; otherwise the manager follows another plan, which leaves the investor at most his outside value before a nonnegative expected payment. Refusal is therefore a best response. If $\underline{\phi}_i < V_i$, some contract under which the manager, serving account i alone, follows the target has an expected payment below V_i, so investor i strictly gains by engaging him alone, and universal refusal is not an equilibrium.

(iii) Under nonnegative payments the manager follows the target only if it pays him at least as much as the deviation, which by Proposition 44 applied to account i's records requires an expected payment of at least $c/(1-\varrho_i)$, and only if he accepts, which requires an expected payment of at least c_{e^*}; paying slightly more than the larger of the two on a record that minimizes the likelihood ratio makes the target his strict choice and secures his acceptance. When the deviation costs nothing, $c = c_{e^*}$ and the larger is $c/(1-\varrho_i)$. When payments may be negative, the manager accepts only contracts whose expected payment under the plan he follows covers its cost, so $\underline{\phi}_i$ is at least the cost of the target's experiment. If own-account compensation of account i strictly implements the target, adding a constant to it changes no comparison among plans, and a constant that brings its expected payment under the target to just above that cost preserves both strict implementation and the manager's acceptance. □

The economy of Appendix B.6

An investor's increment from two lots of program A is $20 + 20\epsilon(\eta - \sigma_i)$, which equals 20 exactly when $\eta = \sigma_i$, and from two lots of B it equals 20 exactly when $\eta = -\sigma_i$; without knowledge of η, or with knowledge of only one quality bit, each event has probability one half. Allocation P makes every increment equal to 20 when $\eta = 1$ and Q when $\eta = -1$. The menu of separate permissions consists of the feasible combinations of none, two lots of A and two lots of B for each investor; a program forms only with at least four lots, so a program held by fewer than two investors cannot be chosen, and 29 combinations remain. Under own-account payments of k for each account that holds a program and whose increment equals 20, a combination in which j investors hold programs pays an uninformed manager $jk/2$, at most $2k$, while the target pays $4k$ and costs 2; learning one bit leaves η uniform and costs 1.2. Investigation is therefore optimal on that menu if and only if $4k - 2 \ge 2k$, and enumeration of all 120 feasible allocations confirms that at $k = 5/4$ it remains strictly optimal among them. The modular identity (21) holds because D combines investors 1 and 2's positions in P with investors 3 and 4's in Q and D' combines the complementary positions, so a sum of account-by-account payments assigns the two pairs the same total in every state. Under the target each investor's terminal increment is 20 for sure, while his outside increment is $-20\sigma_i\epsilon$, whose certainty equivalent is $-\rho_i \log\cosh(20/\rho_i)$ for risk tolerance ρ_i; summing $20 + \rho_i \log\cosh(20/\rho_i)$ over tolerances $(20, 30, 40, 50)$ gives 103.60. Under D, investors 1 and 2 hold an increment equal to 20 with probability one half and to 20 ± 40 otherwise, investors 3 and 4 the mirror image, and the gross certainty-equivalent gain is 56.40, less 7 of fees and 1 of implementation expense. □

The divisible-capital example of Appendix B.11.1

Let $\bar{u} = 9.5$. When the program forms, $Q \le 8$ and its net value per reserved unit is $14.5 - 40/Q \le \bar{u}$; a failed reservation costs 0.5 per unit. At $c = 10$ the separate net returns $(8, -8, -7, -6)$ are all below $\bar{u}$, so every investor's payoff is at most $2\bar{u} = 19$ in every profile, and the all-common profile gives each investor exactly 19. It is therefore an equilibrium that weakly Pareto-dominates every profile, which gives completion and $H = 76$. The all-outside profile is an equilibrium because no investor can reach seven units alone, and its welfare is $L = 16$.

At $c = 7$, consider investor 1. If the program forms with $q_1 > 0$, his payoff is at most $11(2 - q_1) + vq_1 = 22 - q_1(11 - v) < 22$; if it fails, a positive reservation costs him $0.5q_1$ and forgoes $11q_1$. Hence $q_1 = 0$, $x_1 = 2$ is strictly dominant. The others then hold at most six units and cannot form the program, their separate returns are negative, and any reservation is costly, so each holds cash. The unique equilibrium yields 22. In any formed profile the other investors' separate holdings lose money and investor 1's separate holding is at most $2 - q_1 \le 8 - Q$, so welfare is at most $14.5Q - 40 + 11(8 - Q) \le 76$, with equality at all-common; profiles without formation yield at most 22. Hence $V = 76$. After the change investor 1's payoff never exceeds 22 and every other investor's never exceeds 19, so every acceptable incremental vector is dominated by $(3, 19, 19, 19)$, the all-common vector before the change.

Now let the other three investors reserve two units each. Investor 1's payoff from reserving $q \in [1, 2]$, keeping the program formed and investing the rest separately, is $11(2-q)+q[14.5-40/(6+q)]$ plus any transfer. With a flat transfer of 3 conditional on formation, this is 22 at $q = 2$ and 22.79 at $q = 1$, so he reduces his reservation to one unit; the program stays formed, and welfare, in which the transfer nets out, is $(11 + 8.786) + 3 \times 17.571 = 72.5$. With a rebate of r per reserved unit, his payoff is $22 + (r - 1.5)q + 5q - 40q/(6 + q)$, whose derivative in q is $r + 3.5 - 240/(6 + q)^2 > 0$ on $[1, 2]$ for $r \ge 1.5$; a rebate just above 1.5 therefore makes the full reservation strictly optimal and gives him more than his outside value of 22. Under a size-tiered or universal equal-treatment clause with equal commitments, every investor receives the same rebate, at a total cost of $4 \times 3 = 12$. □

Proposition 51

Fix investor i and order the other subscribers by inclusion. His return to subscribing is $-h_i$ while his subscription would fail and $u_i(S \cup \{i\}) - o_i$ once it forms the program. It is constant on the first region and nondecreasing on the second by (24). Every passage from the first region to the second passes through a membership $T \ni i$ with $|T| = \nu$. If (25) fails at such a T, let $S' = T \setminus \{i\}$ and remove any one investor from S' to obtain $S \subset S'$. The return is $-h_i$ at S and $u_i(T) - o_i < -h_i$ at S', so it falls along two ordered profiles. If (25) holds at every such T, each passage is nondecreasing, and the return is nondecreasing throughout. □

Theorem 52

(i) Indispensability implies that every formed profile has $q_i > 0$ for all i and that a refusal by any investor destroys formation, so refusing yields o_i and a formed equilibrium requires $f_i(q_i) \geq 0$. In a failed profile with some $q_i > 0$, reducing q_i to zero saves $\rho_i q_i > 0$, so all-zero is the only failed equilibrium; it is an equilibrium because no investor can form the program alone. At a formed profile with $\sum_i q_i > K$, an investor away from his unique maximizer can move a small distance toward it without destroying formation, so $q = d$ is necessary; it is sufficient when every peak is nonnegative, since no formed deviation beats a peak and a failed deviation yields at most o_i. At a profile with $\sum_i q_i = K$, an investor with $q_i < d_i$ gains by a small increase. If $q_i \geq d_i$, every increase weakly lowers his formed value and every decrease destroys formation, so acceptability suffices.

(ii) If some peak is negative, no formed profile is acceptable for that investor. Otherwise each investor's acceptable upper branch is $[d_i, R_i]$. If $\sum_i d_i \geq K$, the vector d is an equilibrium. If $\sum_i d_i < K \leq \sum_i R_i$, continuity yields quantities in these intervals summing to K, which form an equilibrium by (i). If $\sum_i R_i < K$, a threshold equilibrium would require some $q_i > R_i$ and hence $f_i(q_i) < 0$, and $\sum_i d_i < K$ rules out the slack case.

(iii) An acceptable failed profile has $-\rho_i q_i \geq 0$ for all i, so it is the all-zero equilibrium. Take an acceptable formed allocation q and raise every $q_i < d_i$ to d_i; the program stays formed and each owner weakly gains. If $\sum_i d_i \geq K$, lower every remaining $q_i > d_i$ to d_i; the result is d, an equilibrium by (i), and each investor weakly gains because moving toward the maximizer of a concave function weakly raises its value. Otherwise lower the quantities above their peaks continuously toward those peaks until the total equals K, which is possible because the total is at least K before the reduction and below K at the peaks. Every quantity is then at least its peak, the total is K, and every value has weakly risen, so the profile is an equilibrium by (i) that weakly dominates q.

(iv) Each set $\{q_i : f_i(q_i) \geq 0\}$ is an interval because f_i is concave, so the acceptable formed domain is compact and convex, and the strictly concave objective $\sum_i f_i(q_i)$ has a unique maximizer q° on it when it is nonempty. By (iii), an equilibrium q^* satisfies $f_i(q_i^*) \geq f_i(q_i^\circ)$ for all i; it is acceptable and formed, so it also maximizes the objective and equals q°. Equation (27) is Theorem 31 with the all-zero profile supplying the zero. □

Corollary 53

Assumptions 1 and 2 hold before and after the change, completion holds before it by Theorem 52, and the pointwise order on f_i, with reservation costs unchanged, gives a pointwise decline of incremental payoffs on every profile. Theorem 32 applies. □

Proposition 54

Completion. With outside values zero, no failure costs and threshold two, each pair pays its members 10 and the full program pays each member 1. Pairs are not equilibria, because the

excluded investor gains 1 by entering. The full program is an equilibrium, because a member who leaves turns it into a pair in which he receives nothing. A profile with a single subscriber is not an equilibrium, because another investor gains 10 by joining, so the all-outside profile is the only failed equilibrium and its welfare is zero. Hence $L = 0$, $H = 3$ and $G = 3$, although every pair is acceptable with surplus 20, so completion fails. When investor 3's outside value rises to 1.5, he leaves the full program, the pair $\{1, 2\}$ becomes an equilibrium with welfare 21.5, $L = 1.5$ and $G = 20$. Contraction holds, since only investor 3's incremental payoffs fall.

Contraction. With both investors required and formed payoffs $(10, 10)$, $G = 20$. After investor 1's outside value rises to 1 and formed payoffs rise to $(12, 12)$, $L = 1$, $H = 24$ and $G = 23$. Investor 1's incremental payoff from formation rises from 10 to 11, so the acceptable payoffs after the change are not dominated by those before it.

Independent outside options. If both investors stay out, each receives 5 from the shared outside asset; a lone outsider receives 10, which is each investor's outside value; the program pays 6 to each. A member who leaves the program receives 10, so the program is not an equilibrium. A lone subscriber receives nothing while the other takes the whole asset, so he prefers to stay out. Staying out together is therefore the only equilibrium, with welfare $10 < \sum_i o_i = 20$. Assumption 2 holds, and Lemma 2 fails.

No failure reward. A lone subscriber receives 1. Staying out together is not an equilibrium, since either investor gains 1 by subscribing alone, and joint subscription yields -2 each, so the equilibria are the two profiles with one subscriber, each with welfare 1. □

Proposition 55

Lemma 2 at o' and contraction give $G(o') \leq \sup_{\mathcal{A}(o')} \sum_i g_i(a; o') \leq \sup_{\mathcal{A}(o)} \sum_i g_i(a; o) = W^*(o) - L(o) = G(o) + U(o)$, and $H(o') = L(o) + \Delta + G(o')$. Attainment is shown in the congestion game of Table 6 in Appendix B.14, where $W^*(o) = 20$, $H(o) = 3$, $\Delta = 1.5$ and $G(o') = 20$. □

The example of Appendix B.9.1

Let commitments q form the program, so $Q = \sum_i q_i \in [7, 8]$, and let every committed unit receive a rebate r. Investor 1 earns $14.5 - 40/Q + r$ per committed unit and 11 per unit invested separately, so with $0 < q_1 \leq 2$ his payoff is $22 + q_1(r + 3.5 - 40/Q)$, which reaches his outside value of 22 only if $r \geq 40/Q - 3.5 > 0$; and $q_1 > 0$ is needed, since the others hold at most six units. Under universal equal treatment the cost of securing investor 1 is therefore at least $Q(40/Q - 3.5) = 40 - 3.5Q \geq 12$, and a rate just above 1.5 at $q = (2, 2, 2, 2)$ approaches this bound while making the full reservation strictly optimal, as shown above. If one of investors 2 to 4 waives, at most two units are excluded, and the cost is at least $(Q - 2)(40/Q - 3.5) = 47 - 3.5Q - 80/Q$, whose derivative $-3.5 + 80/Q^2$ is negative on $[7, 8]$, so the cost is at least 9, attained in the limit at $q = (2, 2, 2, 2)$. With two waivers at $q = (2, 2, 2, 2)$ the cost approaches $4 \times 1.5 = 6$. In the notation of Proposition 48, $z = 3$, $\Phi = 8$,

$\xi_j = 3$ for $j = 2, 3, 4$, the deficit is 4, and the minimal covering sets are the three pairs. Investor 1 is pivotal, because without him the others hold at most six units.

Under a size-tiered clause with $q = (2, 1.9, 1.9, 1.9)$, $Q = 7.7$ and only investor 1 is entitled to his terms. He obtains $2(14.5 - 40/7.7) \approx 18.61$ before a concession, so securing him requires a rebate of about 1.695 per unit, costing about 3.39, and each other investor obtains $1.9(14.5 - 40/7.7) \approx 17.68 > 0$. At that rate investor 1's payoff from reserving $q \in [1.3, 2]$, $11(2 - q) + q[14.5 - 40/(5.7 + q) + 1.695]$, is maximized at $q = 2$, where it equals 22. If investor 2 instead restores his ticket to two units, $Q = 7.8$, he becomes entitled to investor 1's terms, the required rate is $40/7.8 - 3.5 \approx 1.628$, his payoff is $2(14.5 - 40/7.8 + 1.628) = 22$, and securing investor 1 costs about $4 \times 1.628 \approx 6.51$; if two of investors 2 to 4 do so, $Q = 7.9$ and the cost is about $6 \times 1.563 \approx 9.38 > 8$. □

Proposition 49

With a common payment b, the set of strictly assured investors is $C = \{i : h_i < b\}$. By the characterization in Theorem 3, which holds for either kind of payment, formation is implemented if and only if some $F \in \mathcal{F}$ has at most one member outside C. Since no singleton forms the investment, every such F has at least two members, and the condition holds for F exactly when b exceeds the second-largest expense in F, attained by leaving out the most expensive member; minimizing over F gives $b > h^\circ$. An escrowed common payment creates a claim $b|S|$ for a failing coalition S of committed investors, so its liability is $b\,\bar{m}$, whose infimum over implementing payments is $\bar{m}\,h^\circ$. Let F° attain h° and let F' be an inclusion-minimal coalition of $\mathcal{F}$ contained in it. Removing members cannot raise the second-largest expense, so that of F' is at most h°, and by the definition of h° it equals h°. Removing the most expensive member of F' leaves a failing coalition of at most $\bar{m}$ members, each with expense at most h°, so $R^* \le \sum_{F'} h - \max_{F'} h \le \bar{m}\,h^\circ$. A common own payment is paid to every investor, all of whom commit in equilibrium, so it pays more than $n\,h^\circ$; if members of the class financed it, each would contribute the same amount, and the net payment to each would be zero, which assures nobody. □

Theorem 36

Sufficiency. An investor with $\beta_i > h_i$ strictly prefers signing whatever others do: if everyone else signs he gains $D_i(\vartheta_i) > 0$, and otherwise he receives $\beta_i - h_i > 0$. At most one investor lacks this property, and he faces universal signing by all others, against which signing yields $D_i(\vartheta_i) > 0$. In any equilibrium, therefore, all sign at every type.

Necessity. If some $D_i(\vartheta_i) < 0$, universal signing is not an equilibrium; since the finite Bayesian game has a mixed equilibrium, universal signing does not hold in every equilibrium. If at least two investors have $\beta_i \le h_i$, let all of them refuse at every type and all others sign. Each refuser faces another refuser, so signing cannot complete the cohort and yields no positive gain; each signer has $\beta_i > h_i$ and strictly prefers signing. This profile is an equilibrium with failure.

It remains to treat $D_i(\vartheta_i) \geq 0$ for all types, at most one investor j with $\beta_j \leq h_j$, and $D_i(\vartheta_i) = 0$ at some type. If j has a type with $D_j = 0$, let those types of j refuse and his other types sign, and let all other investors sign. Investor j is optimizing because everyone else signs. Each other investor's gain from signing is a convex combination of a nonnegative D_i and a strictly positive strategic-failure gain, so signing remains optimal. Otherwise every type of j has $D_j > 0$, and the zero occurs at a type of an insured investor i. Let that type refuse with probability $\varepsilon > 0$ and every other type of every investor sign. Investor i is indifferent at that type because everyone else signs. Every other insured investor remains willing by the convex-combination argument. Conditional on any type of j, the probability of refusal by others is at most ε, so for ε small enough every type of j, whose D_j values are finitely many and strictly positive, still strictly prefers signing. If no investor is uninsured, the last step is unnecessary. The resulting mixed equilibrium fails with positive probability, and correlation among types changes conditional probabilities without affecting the bound ε. □

Propositions 37 and 39

Strict positivity of (17) is equivalent to

$$p\, e^{-\alpha_i(y_i(t_i)+T_i)} < e^{-\alpha_i h_i} - (1-p)e^{-\alpha_i \gamma_i}.$$

The left side is strictly positive at every finite transfer, so a nonpositive right side rules out strict willingness. With a positive right side, taking logarithms and requiring the inequality at the worst type gives (18).

For Corollary 38, balanced transfers with $T_i > r_i$ for every i exist if and only if $\sum_i r_i < 0$: if the sum is negative, set $T_i = r_i - \sum_j r_j/m$; if not, $\sum_i T_i > \sum_i r_i \geq 0$ contradicts balance. Under risk neutrality with $\gamma_i = 0$, $r_i = h_i/p - y_i^{\min}$, which gives (19). Under CARA the condition is $\sum_i r_i < 0$ with every floor finite. At a transfer that leaves investor i indifferent under risk neutrality with $h_i > 0$ and $0 < p < 1$, his payment lottery is nondegenerate, and its certainty equivalent lies strictly below its mean, so his CARA floor strictly exceeds the risk-neutral one; with $h_i = 0$ the two floors coincide.

For the reserve, Theorem 36 requires at least $m - 1$ investors with $\beta_i > h_i$. Those investors may all sign while the remaining investor refuses, a strategic failure in which the liability is $\sum \beta_i > \sum h_i$ over the insured investors, which is at least $\sum_i h_i - \max_i h_i$. Paying $h_i + \varepsilon$ to every investor except one with the highest cost, and nothing to him, implements universal subscription, and its largest liability, incurred when all insured investors sign and the uninsured one refuses, converges to the bound as $\varepsilon \to 0$. At $\varepsilon = 0$ two investors would be uninsured and the refusal equilibrium would return, so the infimum is not attained. □

Proposition 40

A strategic failure is a profile in which a nonempty set of investors refuses, and payments go to signers, so the liability in such a profile is the sum of the signers' payments. With nonnegative payments it is largest when exactly one investor refuses and that investor has the smallest payment, which gives $\sum_i \beta_i - \min_i \beta_i$. If $\beta_j \le \min_{i \ne j} \beta_i$, the largest liability is $\sum_{i \ne j} \beta_i$, which does not depend on β_j. Raising β_j while it remains at most h_j keeps investor j uninsured, so the conditions of Theorem 36 continue to hold, and it raises his strategic-failure utility gain e_j. His gain from signing at belief Q that all others sign is $QD_j + (1 - Q)e_j$, so the threshold belief $-e_j/(D_j - e_j)$ falls. If every investor is strictly insured, the largest liability is the sum of all payments but the smallest; each payment exceeds its investor's cost, so this sum exceeds the sum of all costs but that of the investor with the smallest cost, $\sum_i h_i - \min_i h_i$, and paying $h_i + \varepsilon$ to everyone approaches the bound. If $\min_{i \ne j} \beta_i > h_j$, raising β_j above h_j insures investor j outright, signing becomes strictly dominant for every investor, and the threshold belief falls to zero, still without enlarging the largest liability. □

B Supplementary results and examples

This appendix collects the results and examples that the main text summarizes, in the order of the sections they supplement. For Section 3: screening under private willingness beyond Theorem 13, the limit of information, and escrow under private information and risk aversion in a closed cohort, which extends Theorem 3 to private gains and a common funding risk. For Section 4: the burden a book places on its investors, expected-externality transfers under binding enrollment, and search on three margins. For Section 5: the economy behind Section 5.1, the two propositions on pooled records and delegation, exposure profiles, and equal treatment with its example, assurance under equal treatment and the routes to repair. For Section 6: threshold programs, divisible contributions, completion and complementarity, continuous portfolios, the boundary of compression, excess separation, technology and collective search, incidence, and the technology economy with its measurement. For Section 7: the sensitivity analysis behind its comparison of costs, and one documented transaction.

B.1 Screening under private willingness: risk aversion and the order of commitments

This subsection extends Theorem 13 to risk-averse investors and settles the order of commitments under private willingness, in the setting of Section 3.5.

For investors with constant absolute risk aversion α_k, the monetary inequalities are replaced by their exponential counterparts: in part (iii) a willing investor k with gain g commits exactly when $p_{>k} e^{-\alpha_k g} + (1 - p_{>k}) e^{\alpha_k (h_k - \beta_k)} < 1$, and in part (ii) the unassured investor's condition becomes $p_{-j} e^{-\alpha_j \underline{g}_j} + (1 - p_{-j}) e^{\alpha_j (h_j - \beta_j)} < 1$ and the premium must satisfy $p_{-i} e^{\alpha_i f_i} + (1 - p_{-i}) e^{-\alpha_i (\beta_i - h_i)} > 1$,

while the dominance steps for assured willing investors compare sure outcomes and are unchanged. The difference can reverse a decision: with $p_{>k} = \frac{1}{2}$, $g = 2$, $h_k = 1$ and $\beta_k = 0$ the monetary condition holds, but an investor with $\alpha_k = 1$ values committing at $-\log\left((e^{-2} + e^{1})/2\right) \approx -0.36$ and refuses, whereas $\beta_k = h_k$ makes committing worthwhile whatever his risk aversion.

Proposition 34 (Waste and the order of commitments)**.** *Under willing formation, and with no investor committing after a refusal, as in every equilibrium when each escrowed payment is below the corresponding expense, the expected expense of standing ready that is wasted on failed attempts is $\sum_{k<n} p_{\leq k}(1 - p_{>k})h_k$ in a public order, which is least when investors commit in increasing order of $p_i h_i/(1 - p_i)$, and $\sum_i p_i(1 - p_{-i})h_i$ with simultaneous commitments, which is more than in any order. If commitments are made in public rounds, some round has two or more members, and the terms consist of charges and escrowed payments none of which exceeds the corresponding expense, some perfect Bayesian equilibrium never forms the investment.*

With two investors, each willing with probability 0.6, a gain of 1 from formation and an expense of 2 if it fails, the first to commit in an order expects $0.6 - 0.4 \times 2 = -0.2$ and declines, although willing formation would add 0.24 in expectation. An escrowed payment strictly between 0.5 and 2 to the first investor restores willing formation at an expected cost approaching 0.12, with 0.48 of expected expenses wasted on failed attempts. With simultaneous commitments, a payment slightly above the expense to one investor, together with a premium upon formation slightly larger than two thirds of the margin by which the payment exceeds the expense, and a payment above a quarter of the expense to the other, reach willing formation whatever the losses of unwilling investors, at an expected net cost approaching 0.60; but willing formation then wastes 0.96 of expenses against a gain of 0.72, so that it destroys value, while the order creates it. The order's advantage lies in the expenses it saves, which are real costs, rather than in the transfers it avoids, and Proposition 34 settles the order of commitments by an interchange argument, under which investors who are likely to be unwilling, or whose expense of standing ready is small, should commit first, so that a failure is discovered before others have spent much in waiting for it, while Halac, Kremer and Winter (2020), whose investors differ in size rather than in willingness, find that large investors should be locked in first. The proposition's second statement qualifies the calendar: unless some escrowed payment exceeds its expense, the screening of Theorem 13(iii) needs one investor to a round. Under risk neutrality an own payment of equal expected value would serve the first investor in an order as well as an escrowed one, but under risk aversion the escrow is cheaper, because moving a payment of equal expected value from the state in which the investment forms to the state in which it fails raises the expected utility of a risk-averse investor so long as the failure state remains the worse of the two.

Payments to early contributors when a campaign fails are the instrument of Cason, Tabarrok and Zubrickas (2021), who show that early refund bonuses remove inefficient equilibria of a dynamic assurance contract with private values and observe that restricting bonuses to early contributions lowers the largest amount a campaign might have to pay; Theorem 13 adds that within an order such payments need not exceed the expenses they insure, that a charge upon

formation or commitment achieves the screening without an order, and that without either no terms paid to committing investors achieve it. Independent willingness rules out the cascades that make early commitments informative in sequential offerings and crowdfunding (Welch, 1992; Cong and Xiao, 2024; Deb, Öry and Williams, 2024), and where willingness is correlated an order also transmits information, which can make early commitments more valuable or later ones more fragile. Private-equity funds also screen investors on other dimensions, as when restrictions on transferring fund interests select investors unlikely to need liquidity (Lerner and Schoar, 2004); the screening here is on willingness to form a particular investment, and its instruments are payments in the state of failure and charges in the state of formation.

B.2 The limit of information

Uncertainty about other investors' willingness differs from uncertainty about whether a formed investment pays. Consider finitely many investors whose private types, drawn from a common prior, specify outside values and possibly risk preferences, and who observe their own types together with whatever signals an information structure provides.

Proposition 35 (Information alone)**.** *If no investor can form the program alone and a failed subscription yields no positive net reward, universal refusal is a Bayesian equilibrium under every information structure.*

If everyone else refuses at every type, a subscription cannot form the program and cannot raise the subscriber's payoff, whatever he knows. Greater signal precision, public disclosure of willingness and recorded expressions of interest therefore leave the refusal equilibrium intact. Global-games arguments select a unique equilibrium by adding dominance regions in which some investors act regardless of others (Morris and Shin, 2003; Sákovics and Steiner, 2012). A fixed-threshold financial program has no such region, because a single investor cannot form it whatever he learns, so uniqueness must come from a change in incentives or in the order of moves. Sequential subscription without binding commitments raises different issues (Admati and Perry, 1991; Marx and Matthews, 2000); the proposition concerns simultaneous, one-shot commitments.

B.3 Escrow under private information and risk aversion

Theorem 3 assumes that every investor is known to gain from formation. When each investor knows only his own gain, the escrowed term must be combined with terms that secure participation at every type, and a common funding risk and risk aversion change what those terms can achieve. Fix a cohort M of $m \geq 2$ investors with positive tickets q_i and a minimum K, and let the cohort be minimal winning:

$$\sum_{i \in M} q_i \geq K, \qquad \sum_{j \in M \setminus \{i\}} q_j < K \quad \text{for every } i \in M. \tag{15}$$

Each investor signs or refuses a public contract, and signing binds him to his ticket if the contract's conditions are met. No type is reported. Types are finite, every own type has positive probability,

and the prior may be correlated. Values are private: conditional on his own type, an investor's payoff from full subscription does not depend on others' types.

If all sign, a funding attempt follows. A separately verifiable condition Z, such as delivery of committed cash, succeeds with probability $p > 0$, independently of types. A signer bears an expense of standing ready $h_i \geq 0$ in every state. If the cohort fails to sign he receives $\beta_i \geq 0$; if all sign but Z fails he receives $\gamma_i \geq 0$; and on success he receives $y_i(\vartheta_i) + T_i$, where $y_i(\vartheta_i)$ is his success-state gain over his type's outside portfolio and T_i is a fixed success transfer. A non-signer keeps his outside option. Strategic failure and funding failure are observable contractual events, defined without reference to types.

Let $D_i(\vartheta_i)$ be the expected-utility gain from signing when all others sign, integrating over Z, and let e_i be the utility gain from the sure strategic-failure receipt $\beta_i - h_i$, which has the sign of $\beta_i - h_i$. At any belief, the gain from signing is a probability-weighted average of D_i and e_i; the timing, the independent funding event and the private-values restriction are imposed to obtain this reduction.

Theorem 36 (Bayesian subscription)**.** *In the closed-cohort game, every investor signs at every type in every Bayesian Nash equilibrium, mixed equilibria included, if and only if*

$$D_i(\vartheta_i) > 0 \quad \textit{for every } i \textit{ and } \vartheta_i, \qquad \textit{and} \qquad \#\{i \in M : \beta_i \leq h_i\} \leq 1. \tag{16}$$

The first condition is participation: every type strictly prefers full subscription to his outside portfolio. The second is selection: all but at most one investor are insured against strategic failure by an escrowed payment exceeding his expense of standing ready. For sufficiency, the insured investors strictly prefer signing whether or not the others sign, so at least $m - 1$ investors sign at every type, and the remaining investor then strictly prefers signing because $D_i > 0$. For necessity, a negative D_i defeats universal signing, and two uninsured investors can refuse together, neither able to complete the cohort alone. When some type has $D_i = 0$, that type can mix, and sufficiently rare refusal leaves everyone else willing. Appendix A gives the complete argument, including correlated types.

The strict inequalities in (16) cannot be weakened, because a contract can have a unique, fully subscribed pure equilibrium and also a mixed equilibrium with incomplete subscription. With two investors whose full-subscription gains are $(0, 1)$ and whose strategic-failure gains are $(1, -1)$, universal signing is the only pure equilibrium, yet the first investor signing with any probability in $[1/2, 1)$ while the second always signs is also an equilibrium. A certification based on pure equilibria alone would wrongly declare implementation universal.

Theorem 36 also separates instruments that are routinely conflated, since a binding commitment prevents an accepted ticket from being revised without making anyone accept it, formation-contingent transfers can make full subscription attractive without removing refusal, and assurance can remove refusal without making an unattractive investment acceptable, and the theorem states when both margins have been addressed.

B.3.1 The price of participation

Under CARA preferences over a payment lottery independent of residual background wealth, with coefficient $\alpha_i > 0$ fixed across investor i's types, his certainty-equivalent gain from signing when all others sign is

$$\Upsilon_i(\vartheta_i) = -\frac{1}{\alpha_i}\log\Big[p\, e^{-\alpha_i(y_i(\vartheta_i)+T_i)} + (1-p)\, e^{-\alpha_i\gamma_i}\Big] - h_i, \tag{17}$$

whose sign is the sign of $D_i(\vartheta_i)$. Write $y_i^{\min} = \min_{\vartheta_i} y_i(\vartheta_i)$.

Proposition 37 (Participation floor)**.** *If $e^{-\alpha_i h_i} - (1-p)e^{-\alpha_i\gamma_i} > 0$, every type of investor i strictly gains from full subscription if and only if*

$$T_i > r_i \equiv -y_i^{\min} - \frac{1}{\alpha_i}\log\left(\frac{e^{-\alpha_i h_i} - (1-p)e^{-\alpha_i\gamma_i}}{p}\right). \tag{18}$$

If $e^{-\alpha_i h_i} - (1-p)e^{-\alpha_i\gamma_i} \le 0$, no finite success transfer makes any type strictly willing.

The impossibility has an economic source: a payment in the success state cannot offset the utility cost of a sufficiently likely uncompensated failure. With $p = 0.1$, $\alpha_i = 1$, $h_i = 1$ and $\gamma_i = 0$, the certainty-equivalent gain is bounded above by $-1 - \log 0.9 < 0$ however large the success transfer. As $\alpha_i \to 0$ the floor converges to $[h_i - (1-p)\gamma_i]/p - y_i^{\min}$. With privately known risk aversion the floor is computed type by type and its maximum taken. Balanced transfers that clear every floor exist if and only if a finite linear program has a strictly positive value (Internet Appendix I). Balance in the success state does not make the certainty-equivalent effects cancel, because success transfers also reallocate risk.

The floors explain why private outside values make the distribution gap hard to close. A transfer schedule that does not depend on reports must satisfy every type, so it is priced for each investor's best alternative.

Corollary 38 (Worst-case pricing)**.** *Let $\gamma_i = 0$ and let success transfers be balanced and otherwise unrestricted. Under risk neutrality, some transfer vector makes every type of every investor strictly willing if and only if*

$$p\sum_{i\in M} y_i^{\min} \;>\; \sum_{i\in M} h_i. \tag{19}$$

Under CARA preferences the condition is $\sum_i r_i < 0$ with every r_i from (18) finite, which is at least as demanding, and strictly so when $p < 1$ and some $h_i > 0$.

Under risk neutrality, the program must create surplus at the profile in which every investor draws his worst type at once, and under risk aversion the requirement is stricter. With independent types that profile can be rare while formation is efficient at most realizations. With correlated types it need not occur at all, because the investors' individually worst types can lie outside the joint support, and fixed terms must still accommodate each investor's worst case as though all could coincide. The cost of the restriction is then $p[\min_\vartheta \sum_i y_i(\vartheta_i) - \sum_i y_i^{\min}]$, the difference

between the worst profile that can arise, the minimum taken over the support, and the sum of the marginal worst cases. Whenever (19) fails, the efficient program cannot be supported by type-independent terms, however rarely the worst profile arises. Allowing investors to report their types relaxes the restriction, and what it achieves depends on the information structure. With correlated types, report-dependent transfers can generally restore efficient, individually rational and budget-balanced participation (Crémer and McLean, 1988; Kosenok and Severinov, 2008), although the transfers they require can be large, and limited liability can rule them out. With independent private values the difficulty can remain: in the environment of Mailath and Postlewaite (1990), where the cost of provision grows in proportion to the number of participants, incentive compatibility, voluntary participation and budget balance cannot be reconciled with efficient provision, and the probability of provision vanishes as the number of participants grows; with a fixed cost of provision the asymptotics differ. Corollary 38 makes the narrower claim, which does not depend on either environment: terms that do not depend on reports and must satisfy every type's participation constraint are priced for every investor's worst type at once.

B.3.2 The resources behind assurance

Because assurance is a promise, it must be backed by a reserve, and under private values and a common funding risk the smallest such reserve keeps the form of Theorem 3, with $\mathcal{F} = \{M\}$.

Proposition 39 (Assurance reserve)**.** *Suppose every type strictly gains from full subscription. Within the class of nonnegative strategic-failure payments that implement universal subscription in every equilibrium, the infimum of the largest liability that strategic failure can create is*

$$R_{\text{sel}}^{\inf} = \sum_{i \in M} h_i - \max_{i \in M} h_i, \tag{20}$$

and it is not attained.

Leave an investor with the highest expense of standing ready uninsured, pay every other investor $h_i + \varepsilon$, and let ε fall to zero; those $m - 1$ payments fall due together if the remaining investor refuses. Theorem 36 requires at least $m - 1$ strict assurances, which gives necessity, and strictness is why the bound is an infimum: exact equality restores a refusal equilibrium.

The largest liability has a simple form that the infimum does not exhaust. Because strategic failure requires at least one refusal and payments go only to signers, a schedule of nonnegative payments β creates a largest liability of $\sum_i \beta_i - \min_i \beta_i$. The uninsured investor's payment can therefore be raised to the smallest payment received by any other investor without enlarging the reserve.

Proposition 40 (Reserve-neutral protection)**.** *Let β implement universal subscription with uninsured investor j. Raising β_j to any level not exceeding $\min_{i \neq j} \beta_i$ leaves the largest strategic-failure liability unchanged and lowers the probability that investor j must assign to universal signing by the others before he is willing to sign; if that level exceeds h_j, investor j becomes insured outright.*

If every investor is to be strictly insured, so that signing is strictly dominant for all, the infimum of the largest liability is $\sum_i h_i - \min_i h_i$*, which exceeds (20) by* $\max_i h_i - \min_i h_i$.

The extra reserve required to remove the last investor's strategic dependence is thus governed by the dispersion of expenses of standing ready, and it vanishes when those costs are equal. The exact equilibrium analysis is indifferent to reserve-neutral protection, since the same contract implements universal subscription with or without it, but the protected contract tolerates more pessimism about the others' behavior at no cost in reserve capacity, though possibly at a higher expected payout. When universal subscription is implemented the reserve is never drawn, but holding it ready has a carrying cost. A promise γ_i upon funding failure falls due only after every investor has signed, an event exclusive of strategic failure, so a single reserve equal to the larger of the two liabilities covers both, provided the same funds may legally serve at the relevant dates.

Neither promise pays for the investment itself. A cash backstop needs capacity equal to the largest shortfall, across the settlement states it covers, of collectible committed funding below required cash. Its expected payout averages the shortfalls; its economic price values them at the provider's state prices, which are highest when shortfalls coincide with scarce aggregate liquidity (Holmström and Tirole, 1998); and its carrying cost prices capital held ready. Cash cannot cure a missing consent or an ineligible owner. Table 4 collects the distinctions.

Table 4: Five failures and the promises that repair them

Failure	Instrument	Requirement
Selection	Escrowed payment for strategic failure	Every type strictly gains from full subscription, and all but at most one indispensable investor receive a payment exceeding their expense of standing ready (Theorem 36).
Participation	Success-contingent transfer	The worst-type floor (18) is finite, and the floors are jointly compatible with transfer limits and budget balance (Proposition 37).
Cash funding	Prefunding, escrow or collectible backstop	Cash arrives in every covered state; capacity, expected payout and price are different quantities.
Permission	Current authority and satisfied legal conditions	Compensation does not enlarge authority, and cash does not cure a missing consent.
Quantity revision	Binding accepted ticket	A later change of preference cannot rewrite an accepted obligation; revoking authority blocks only new ones.

The first two rows restate Theorem 36 and Proposition 37. The remaining rows state the requirement each instrument must meet; the model does not formalize them.

B.3.3 A finite economy

Four investors with tickets of \$80,000, \$100,000, \$120,000 and \$100,000 face a \$350,000 minimum, so the proposed \$400,000 purchase requires all four. Each investor has two equally likely

outside values, and the high type of investor 4 dislikes success at the initial terms. Success transfers of $(-600, -500, -500, 1{,}600)$ dollars balance and make every type strictly willing, under risk neutrality and under CARA with coefficient 0.02 per thousand dollars. Strategic-failure payments of \$510, \$810 and \$610 to the first three investors, and nothing to the fourth, then implement universal subscription in every equilibrium. The maximum reserve is \$1,930 against an infimum of \$1,900, and carrying it for 90 days at 4 percent costs \$19.04. Paying the fourth investor \$510 as well leaves the maximum reserve at \$1,930, by Proposition 40, and lowers the probability that his high type must assign to universal signing by the others, under risk neutrality, from 79.4 percent to 65.3 percent. Insuring all four strictly would require a reserve above \$2,400. A transfer cap of \$500 per investor cannot support every type; a cap of \$2,000 can. Common funding still fails with probability 0.1, and a backstop that guaranteed the \$400,000 purchase would need the full amount, with an expected draw of \$40,000. The three instruments, transfers for participation, assurance for selection and a backstop for cash, are distinct obligations with distinct prices (Internet Appendix II).

B.4 The burden a book places on its investors, and expected-externality transfers under binding enrollment

This subsection states two results of Section 4.4 in the setting of Theorem 22: the distribution of the total net burden under the leave-one-out rebate of Proposition 24, and the expected-externality transfers of Proposition 25(ii) under binding enrollment, where the veto that defeats them is absent.

Corollary 41 (The burden a book places on its investors)**.** *Under the hypotheses of Proposition 24(ii), the total net burden of the placed investors, $C_n + \Lambda_n$, converges in distribution to $\chi^2_3/(4g_c(0))$, where χ^2_3 is a chi-squared variable with three degrees of freedom, and its expectation converges to $3/(4g_c(0))$. With the premium returned in equal shares the total net burden is C_n, which converges in distribution to $\chi^2_1/(4g_c(0))$.*

Of the three degrees of freedom, one comes from the common imbalance, which fixes how many investors are moved and therefore the real cost of the moves, and the other two come from the gap between the costs of the last two investors moved, which fixes the premium that is not returned and is, in the limit, independent of the imbalance. The leave-one-out rebate therefore triples the expected net burden while giving its distribution a thinner upper tail relative to its mean: the 99th percentile of the limit, $2.84/g_c(0)$, is 3.8 times its mean, against 6.6 times for the cost of the moves alone. The total is not a liability to be reserved against, since the cost of the moves falls on the investors who are moved and the premium that is not returned is revenue to whoever operates the assignment, so that the sum measures what the book takes from its investors, part of it as a transfer, rather than a loss to the economy or a claim on capital outside the book.

Corollary 42 (Expected-externality transfers under binding enrollment)**.** *Under binding enrollment, let the placed investors' transfers be those of Proposition 25(ii) without a fee. Truthful*

reporting is a Bayesian equilibrium among the placed investors, their net burdens sum to C_n in every state, and enrolling exactly when one's margin is at least $\mathbb{E}[C_n]/n$ is a best reply to every enrollment behavior of the others, so that the threshold rule and truthful reports form a Bayesian equilibrium of the game of enrollment and reporting; if $S_n/n \to 0$, Q'_n converges in distribution to a Poisson variable with mean $\mu_v = \gamma/4$.

B.5 Search on three margins

Collective search, the identification of investors whose mandates admit a program and the assembly of their commitments on compatible terms, acts on each of the three quantities, and not always in the same direction.

Proposition 43 (Search, commitment and discretion)**.** *(i) Adding willing investors, or widening permissions without making any investor unwilling, weakly lowers B^* and $R^*(\Gamma, \mathcal{W})$, and lowering any investor's expense of standing ready weakly lowers $R^*(\Gamma, \mathcal{W})$.*

(ii) If search leads a share η of narrow mandates to follow a common recommendation, the flexible share required converges to the value in Theorem 18(iii), which is positive and increasing in η; if the recommendation shifts favored projects in the same way and costs satisfy the hypotheses of Theorem 19(iv), the cost of discretion per investor converges to a positive limit.

(iii) If better outside options make some investors unwilling, the formation structure of the remaining book is contained in the original one, so that $R^*(\Gamma, \mathcal{W})$ and B^* weakly rise.

The effect of a technology that searches on behalf of a population therefore depends on whether it diversifies the population's mandates or concentrates them. It lowers the cost of commitment when it finds complementary permissions and lowers the expense of standing ready, and it raises the cost of discretion when it leads many investors to the same recommendation. Search also supplies replacement capital, since a technology that identifies, before the imbalance is known, capital willing to take a vacated place on stated terms lowers the replacement capital that Corollary 23 requires of a book whose investors keep the right to decide later. Better separate accounts act through the third margin, by making investors unwilling, and Section 6 develops that force.

B.6 An economy in which authority appears to matter

This subsection works through the economy that Section 5.1 summarizes, which is built to suggest that authority over combinations of positions is needed, contrary to Corollary 30. Four investors each hold capital for two lots of \$50,000 in two prospective programs, A and B, and each program forms only with at least four lots, so that the feasible set contains 120 joint allocations; amounts below are in thousands of dollars. Two independent fair quality bits determine $\eta = 1$ if they match and $\eta = -1$ otherwise, and an independent fair shock $\epsilon \in \{-1, 1\}$ is never observed

before investment. Each 100 invested returns its principal plus $R_A = 20 + 20\eta\epsilon$ in program A or $R_B = 20 - 20\eta\epsilon$ in program B, and investor i carries a background exposure $W_i = -20\sigma_i\epsilon$ with $\sigma = (1, 1, -1, -1)$, so that with x_{iA} and x_{iB} lots his account's increment is $W_i + \frac{1}{2}x_{iA}R_A + \frac{1}{2}x_{iB}R_B$. The manager can learn neither bit, either one or both, at costs 0, 1.2, 1.2 and 2; investors have CARA risk tolerances $(20, 30, 40, 50)$; and each active program consumes 1 of implementation expense. The target learns both bits and chooses P, which places investors 1 and 2 in program A and investors 3 and 4 in program B, when $\eta = 1$, and the reverse assignment Q when $\eta = -1$. Under the target every account's increment equals 20 in every state, because the program the manager assigns to an investor hedges his background exactly, and the investors' certainty equivalents exceed those of their outside portfolios by 103.60 before fees and expenses.

Suppose each investor grants only a list of the positions he permits in his own account: none, two lots of A, or two lots of B. The menu the manager then faces contains every feasible combination of those lists, 29 allocations, among them D, in which everyone holds program A, and D', in which everyone holds B. Under a compensation schedule tied to allocations, paying 8 when the manager chooses P or Q and every account's increment equals 20, 7 when he chooses D and nothing otherwise, the uninformed manager prefers D, which pays 7, to investigating, which pays $8 - 2 = 6$, so that a joint mandate permitting only $\{0, P, Q\}$, built at a cost of 4, is needed to restore investigation, and it raises the investors' aggregate certainty equivalent from 48.40 to 89.60.

No collection of separate accounts could pay the manager on that schedule, however, and the apparent need for authority disappears with it. Allocation D takes investors 1 and 2's positions from P and investors 3 and 4's from Q, and D' takes the rest, so that any sum of own-account payments satisfies

$$\Psi(D, \omega) + \Psi(D', \omega) = \Psi(P, \omega) + \Psi(Q, \omega) \qquad \text{in every state } \omega, \tag{21}$$

whereas the schedule pays $7 + 0$ against 8. Own-account compensation is modular on the menu of separate permissions, and the schedule that made authority matter is not. Let each investor instead pay the manager k whenever his account holds a program and its increment equals 20. An allocation in which j investors hold programs then pays an uninformed manager $jk/2$ in expectation, at most $2k$, and the informed target pays $4k$, so investigation is optimal on the menu of separate permissions when $k \geq 1$ and strictly so when $k > 1$; with $k = 5/4$ the manager earns 3 by investigating against $5/2$ from the best uninformed allocation, investigation remains his strict choice even among all 120 feasible allocations, and investors pay 5 in expected fees with no mandate to build. The least compensation that induces investigation is 4 whether it is paid account by account or only when all four accounts are right, so the joint mandate's one distinctive feature in this economy is its construction cost. Table 5 collects the comparison.

Table 5: Authority and compensation in the four-investor economy

Menu	Compensation	Investigates	Investors	Manager
Separate permissions (29)	Tied to allocations	No	48.40	7.00
Joint mandate $\{0, P, Q\}$	Tied to allocations	Yes	89.60	6.00
Separate permissions (29)	Own account, $k = 5/4$	Yes	96.60	3.00
Separate permissions (29)	Own account, $k \downarrow 1$	Yes	97.60	2.00

Investors: sum of certainty-equivalent gains over the same outside portfolios, net of fees, implementation expenses and, for the joint mandate, a construction cost of 4. Manager: expected profit after the cost of investigation. Thousands of dollars. The last row is a limit; at $k = 1$ the manager is indifferent.

B.7 Delegation and pooled records: the propositions

This subsection states the two results on delegation that Section 5.1 summarizes, in the setting of Theorem 29. Suppose compensation must be nonnegative, and that the deviation the target must deter is a single plan π, such as not investigating, that saves the manager $c = c_{e^*} - c_\pi > 0$. For a set of records on which compensation may depend, let ϱ be the smallest likelihood ratio between the deviation and the target, $\varrho = \min \mu^\pi(r)/\mu^\tau(r)$ over records with $\mu^\tau(r) > 0$, the minimum taken over joint records when compensation is joint and over pairs of an account and its record when it is own-account.

Proposition 44 (Pooled records)**.** *If $\varrho < 1$, the least expected compensation that makes the target weakly preferred to the deviation is $c/(1-\varrho)$, attained by paying only on a record that minimizes the likelihood ratio; if $\varrho = 1$, the deviation reproduces the target's law on the records available and no finite compensation deters it. The joint ratio never exceeds the smallest marginal ratio, so that when $\varrho_{\text{own}} < 1$ own-account compensation costs $(1-\varrho_{\text{joint}})/(1-\varrho_{\text{own}}) \geq 1$ times as much as joint compensation.*

The pure form of the contract in the example of Section 5.1, which pays only when every one of 20 bets is right, pays about 183,000 times the research cost with probability below 10^{-5} and requires risk neutrality and unbounded payments, so that pay on pooled profit and loss above a hurdle is the practical form; carried interest computed on a whole fund is own-account pay on a diversified account, deal-by-deal carried interest pays on records that pool fewer bets (Litvak, 2009; Hüther et al., 2020), Axelson, Strömberg and Weisbach (2009) study how pooling across deals governs a manager's investment incentives, and in general a scalar return does not reveal everything a joint record contains.

Paying a manager together is itself a formation problem whenever no single account's payment is enough. Let accounts be physically separate, $X = \prod_i X_i$, so that the manager can serve any set of accounts whose owners engage him, and let each investor decide whether to engage him by an own-account contract ϕ_i; the manager accepts an engagement only if his expected compensation

covers the cost of the plan he follows. Call payments *target-maximal* if the target maximizes each account's expected payment among all plans, gross of the cost of investigation, as payment for being right does. Given the set S of engaged accounts, the manager follows a plan that is optimal for $\sum_{i\in S}\phi_i$, breaking ties against investigation; let $\mathcal{F}_\phi$ be the family of sets of engaged accounts for which the target is that plan. Here investors, like the manager, are risk neutral, so that compensation costs them its expected value; with risk-averse investors the comparisons below are between the certainty equivalent of the investigated record net of the compensation lottery and the outside value. Let V_i be investor i's gain from the target over his outside value before payments, suppose that the manager's other plans leave every investor no better off than his outside value, and let $\underline{\phi}_i$ be the infimum of the expected own-account payment over admissible contracts under which the manager, serving account i alone, accepts the engagement and follows the target.

Proposition 45 (Delegation is formation)**.** *(i) With target-maximal payments, $\mathcal{F}_\phi$ is closed under enlargement, and the investors' decisions to engage the manager form a formation game in which an engaged investor receives the target's value net of his payments when the engaged accounts belong to $\mathcal{F}_\phi$, and the value of the manager's other plan net of his payments otherwise.*

(ii) When each investor chooses his contract, universal refusal is an equilibrium if $\underline{\phi}_i \geq V_i$ for every investor, and it is not an equilibrium if $\underline{\phi}_i < V_i$ for some investor, who is then an anchor.

(iii) When payments must be nonnegative and a single deviation binds, $\underline{\phi}_i = \max\{c/(1-\varrho_i),\, c_{e^*}\}$, where ϱ_i is the smallest likelihood ratio of account i's record, which is $c/(1-\varrho_i)$ when the deviation costs nothing; when they may be negative, $\underline{\phi}_i$ equals the cost of the target's experiment whenever own-account compensation of account i alone strictly implements the target in the sense of Theorem 29.

In an economy with four concentrated accounts, each holding a bet that is right with probability 0.6 under investigation and 0.5 without it, a right bet worth 10 and a wrong one costing 10 relative to the owner's outside portfolio, and research costing 1, each investor values investigation at 2, while a lone account would have to pay 6 to motivate it, so that universal refusal is an equilibrium; payment only when all four bets are right lowers the least expected compensation from 6, the least that pay on each account's own record can achieve, to 1296/671, about 1.93, and an account holding a quarter of each bet could attain that amount alone. A fixed research cost that no single investor values above its cost makes delegation a formation problem even with diversified accounts and unlimited liability, as when research costs 1 and each of two investors gains 0.6.

B.8 Records and exposure profiles

Let a book hold m assets for n investors, with exposures $x = (x_{ij})$, investor i's row being his exposure to each asset. A book that offers k *profiles*, the rows of a $k \times m$ matrix P, and lets each

investor hold a combination a_i of them implements $x = AP$, where A has rows a_i. Investors who cannot share a vehicle, because their tax status, regulatory regime or jurisdiction requires a different structure, form T *classes*, each needing vehicles of its own. A book keeps a record of each interest an investor actually holds in a profile and of each position a vehicle actually holds in an asset, at a cost $\kappa > 0$ for each record. When A and P are dense, as the factors of a generic low-rank approximation are, and every class holds the same k profiles through vehicles of its own, the book keeps $nk + Tkm$ records, and this dense benchmark is the one the proposition treats; a pooled fund is its cheapest case, $k = 1$. Vehicles that each hold a single asset, $P = I$, keep instead a record of each position an investor actually holds and one for each vehicle's position, which is cheap when holdings are sparse, as in deal-by-deal syndicates whose investors each hold a few transactions, so that whether common profiles or single-asset vehicles economize more on records depends on whether optimal holdings are close to low rank or close to sparse. Let investor i have constant absolute risk aversion α_i, let asset returns and investors' background wealth be jointly normal, the returns having mean μ and nonsingular covariance Σ, and let σ_i be the covariance of investor i's background wealth with them, so that his optimal exposure is $x_i^* = \alpha_i^{-1}\Sigma^{-1}\mu - \Sigma^{-1}\sigma_i$ and holding x_i instead costs him $\frac{\alpha_i}{2}(x_i - x_i^*)^\top\Sigma(x_i - x_i^*)$ in certainty-equivalent dollars. Let $\nu_1 \geq \nu_2 \geq \dots$ be the singular values of the $m \times n$ matrix whose i-th column is $\sqrt{\alpha_i}\,\Sigma^{1/2}x_i^*$.

Proposition 46 (Records and profiles)**.** *(i) The least aggregate certainty-equivalent loss from restricting investors to combinations of k profiles is $\frac{1}{2}\sum_{j>k}\nu_j^2$ when combinations may be signed, and no smaller when they must be long.*

(ii) In the dense benchmark, for a book that chooses the number of profiles to maximize aggregate certainty equivalents net of the cost of records, the largest optimal number is $k^* = \max\{1, \max\{k : \frac{1}{2}\nu_k^2 \geq \kappa(n + Tm)\}\}$, which weakly rises as κ falls and weakly falls as T rises.

(iii) If every investor's background covariance is proportional to a single vector, the optimal exposures span at most two dimensions, so that two profiles lose nothing; heterogeneous background exposures can raise the rank to one more than the dimension of their span.

Part (i) is the theorem of Eckart and Young (1936) in the metric of the investors' risk-adjusted losses, and the mutual fund theorem is the case $\nu_2 = 0$, in which a single profile serves everyone and pooled exposure is costless. Mamaysky and Spiegel (2001) derive the objectives of funds from the heterogeneous trading needs of investors who cannot trade continuously and let the number of funds respond to the cost of offering them, and Acharya and Bisin (2005) characterize optimal market structures by the dispersion of participants' exposures to the traded assets; the proposition gives the trade-off between the number of profiles and their cost a closed form in the CARA–normal case, as the tail of a spectrum set against the cost of records across classes of investors who cannot share a vehicle. Pooled exposure is in this sense a way of economizing on records rather than a requirement of collective investment. The losses fall on investors whose background wealth, liabilities or concentrated holdings differ from the population's, such as a

founder whose wealth is concentrated in one sector and who should hold private exposure that excludes it, which no common profile can provide. Technology that lowers the cost of forming vehicles, documenting interests, reporting positions and cash flows, and tracking the dependencies that define an investor's class lowers κ and, by part (ii), raises the number of profiles a book should offer, toward an exposure tailored to each investor, while each additional class of investors who cannot share a vehicle adds κm to the cost of every profile. Part (i) bounds what profiles can achieve when combinations are signed; vehicles that may only be held long, positions that cannot be shorted and minimum holdings make the achievable set smaller, so that the bound is a benchmark for measurement rather than a design, and classes that held profiles of their own could improve on the common design that the record count assumes. With the willing set and the expenses of standing ready fixed, neither the cost of records nor the cost of incentive pay raises the cost of commitment support, which by Proposition 17 depends on exposure only through who is willing to commit, although administration that changes either can; and neither raises the discretion a book needs, whose amount depends on permissions (Theorem 18) and whose cost depends on investors' costs of being moved (Theorem 19), which pro rata exposure raises rather than lowers.

B.9 Equal treatment: terms held in common

Investors in shared vehicles rarely pay one another. Their economics are set by the program's terms, such as fees, carried interest, allocations and rebates, and those terms are commonly protected by most-favored-nation clauses, which extend a concession granted to one investor to every investor with an equal or larger commitment and sometimes to all investors. Private equity funds commonly charge investors in the same fund different fee tiers (Begenau and Siriwardane, 2024), and access to the best terms is typically graded by commitment size. Equal treatment protects investors against discrimination, and like the most-favored-customer clauses studied by Cooper (1986), it is also a commitment: it makes a selective concession expensive. It bites at the moment the formation problem operates, because a concession needed to obtain a late, necessary commitment is extended to investors who signed earlier. Under *rate matching*, an entitled investor receives any reduction granted to the investor whose terms he may claim; under *final-terms matching*, he may claim terms no worse than the other investor's, taking account of the terms he already has.

Consider a program being raised, with members S and commitments $q_i > 0$, at a profile that is acceptable with slacks s_i. Every member except k has signed, and his commitment binds if the program closes; k's commitment remains open. Before k signs, his outside value rises by $\Delta > s_k$, so that his shortfall is $z = \Delta - s_k$. A manager or sponsor may grant concessions as reductions $\zeta_i \geq 0$ in a member's per-unit charge χ_i, funded from a rent of at most Φ; concessions are nonnegative because terms already accepted cannot be made worse without consent. For each member j let $\mathcal{S}_j \subseteq S$, with $j \in \mathcal{S}_j$, be the members entitled to j's terms, and let entitlement be transitive. The initial charges honor the clause, and $\omega_j = \chi_k - \chi_j \geq 0$ is the advantage that an entitled member $j \in \mathcal{S}_k$ already holds over k.

Proposition 47 (The cost of equal treatment)**.** *Investor k's commitment can be secured on these terms if and only if $\Phi \geq C_k(z)$, where*

$$C_k(z) = \lambda_k z, \qquad \lambda_k = \frac{\sum_{j \in \mathcal{S}_k} q_j}{q_k} \geq 1, \tag{22}$$

under rate matching, and

$$C_k(z) = \sum_{j \in \mathcal{S}_k} q_j \left[\frac{z}{q_k} - \omega_j \right]_+ \tag{23}$$

under final-terms matching. In both cases $z \leq C_k(z) \leq \lambda_k z$. The final-terms cost is convex and piecewise linear in z, with a kink at $z = q_k \omega_j$ for each entitled member whose terms are better than k's, and it equals $\lambda_k z$ when no entitled member's terms are better than k's.

An investor who is indifferent commits, as acceptability has been defined throughout, so that a guarantee that does not rely on the convention requires $\Phi > C_k(z)$. Equal treatment converts a transfer that the surplus would make affordable into one that can exceed every available rent, and the two forms of matching can differ by large factors that existing fee tiers decide. Let a pivotal investor with a commitment of $20 million pay 1.75 percent and require a reduction to 1.5 percent, and let the only other entitled investor have committed $80 million at 1.25 percent: rate matching extends the quarter-point reduction to both, at a cost of $250,000 a year, while final-terms matching costs $50,000, because the larger investor's charge is already below the pivotal investor's new one. The theory therefore predicts thresholds in the cost of repair at existing fee tiers.

The investors whom a clause protects could waive their entitlements, and a waiver is a separate decision by each of them. Let k be *pivotal*, in that the remaining members cannot form an acceptable program without him, let each member $j \in \mathcal{S}_k \setminus \{k\}$ be able to waive, conditional on formation, every entitlement that arises from the concession to k, and let $\xi_j(z)$ be what his entitlement costs, so that $C_k(z) = z + \sum_j \xi_j(z)$; let $\mathcal{S}_k^+$ be the entitled members whose claims cost something. Waivers are solicited simultaneously before the closing date, each requires a review whose cost $\eta_j \geq 0$ is incurred whether or not the program forms, and the manager then grants the cheapest admissible schedule if it costs at most Φ.

Proposition 48 (Waivers)**.** *Let k be pivotal, let $z \leq \Phi < C_k(z)$, and let $0 \leq \eta_j < s_j$ for every $j \in \mathcal{S}_k^+$. In the waiver game:*

(i) a waiver set $J \subseteq \mathcal{S}_k^+$ rescues the program if and only if $\sum_{j \in J} \xi_j \geq C_k(z) - \Phi$; rescuing pure equilibria exist, and they are exactly the profiles whose waiver set is minimal;

(ii) universal non-waiver is a pure equilibrium if and only if no single waiver covers the deficit, $\max_{j \in \mathcal{S}_k^+} \xi_j < C_k(z) - \Phi$, and it is then a strict equilibrium if every η_j is positive;

(iii) every pure equilibrium rescues the program if and only if $\max_{j \in \mathcal{S}_k^+} \xi_j \geq C_k(z) - \Phi$;

(iv) if waivers are solicited one at a time and each is observed before the next, every subgame-perfect equilibrium rescues the program, and wherever rescue remains possible an entitled member waives exactly when the members solicited after him cannot cover what remains of the deficit.

Renegotiating terms held in common is itself a formation problem. The forgone concessions are contributions, the deficit is the threshold, and a waiver conditional on formation is a contribution refunded on failure, as in the provision-point games of Bagnoli and Lipman (1989), so that universal refusal to waive is an equilibrium whenever entitlement is dispersed, and the rescuing equilibria, in each of which a minimal set gives up what the others keep, are many and unequal. Sequential solicitation removes the refusal equilibrium, as an order does in Section 3.4, and settles the distribution by the order of solicitation; a payment to waiving investors if the waiver round fails plays the role of the escrow. Appendix B.9.1 works through an example in which universal equal treatment turns a repair of 3 into one of at least 12, more than the manager's fee, and records the routes by which the condition $\Phi \geq C_k(z)$ can be met. Equal treatment raises the price of assurance in the same way: when all investors are entitled to the same payment, a common escrowed payment must exceed the smallest, over forming coalitions, of the second-largest expense among their members, and its liability is then at least R^*, rising in the four-investor economy of Appendix B.3.3 from \$1,900 to \$2,400 (Proposition 49 in Appendix B.9.2). Individual terms are thus constrained in the same way as individual commitments, by the need to obtain a collective agreement, while individual exposure is constrained by the cost of records and incentive pay.

B.9.1 An example

Return to the divisible-capital example of Appendix B.11.1 at $c = 7$, and suppose that 8 of the program's cost of 40 is the manager's fee, the rent from which concessions can be funded, and that the program gives every investor the same per-unit terms. Concessions are rebates against each investor's per-unit charge, his share of the program's cost, which is 5 per unit at full commitment. Investors 2 to 4 have signed for two units each, binding if the program closes, and investor 1's two units are needed to reach seven. A per-unit rebate of just over 1.5 on investor 1's two reserved units secures his commitment and, as Appendix B.11.1 shows, also fixes his reservation at two units, so without the clause the repair would cost 3 and would be affordable. Under universal equal treatment $\lambda_1 = 4$, so securing him costs 12, more than the manager's entire fee, and the program fails although it would produce 54 above the outside floor.

The failure does not rest on equal commitments. Investor 1 invests separately whatever he does not commit, at 11 per unit, so the rebate rate needed to give him his outside value of 22 is $40/Q - 3.5$ whatever his own commitment, and under universal equal treatment its cost, Q times that rate, is at least $40 - 3.5Q \geq 12$ for every commitment profile that forms the program (Appendix A). A size-tiered clause behaves differently. If investors 2 to 4 agree to amend their signed commitments to 1.9 units each, the program still forms with 7.7 units, investor 1 becomes uniquely largest and the only investor entitled to his terms, and securing him costs about 3.39

while each of the others still gains about 17.68. A size-tiered clause can therefore be escaped by restructuring tickets, provided the manager can cap allocations at 1.9 units. Without caps, each of investors 2 to 4 gains by restoring his ticket to two units, which entitles him to investor 1's rebate and raises his payoff from about 17.68 to 22. One such increase is still affordable, at a cost of about 6.51, but two raise the cost to about 9.38, more than the fee, so uncapped resizing becomes a coordination problem among the investors it excludes. Where size-tiered clauses govern, rescues should come with resized and capped commitments.

Under universal equal treatment, waivers do not reliably rescue the program either. Investor 1 is pivotal, each of investors 2 to 4 holds an entitlement worth 3, and the deficit is $12 - 8 = 4$. One waiver lowers the cost to 9, still above the fee; two lower it to 6. By Proposition 48, universal non-waiver is an equilibrium, strict if a waiver carries any review cost, each pair of waivers is a rescuing equilibrium, and in each rescue the third investor keeps a rebate of 3 that the other two forgo. Redesigning commitments does not let a single waiver suffice, because with one waiver the cost of securing investor 1 is at least 9 under every commitment profile that forms the program (Appendix A). The program survives only if two of the three investors coordinate on giving up what the third retains.

B.9.2 Assurance under equal treatment

Both constructions in Theorem 3 discriminate, since they assure some members of a forming coalition and not the most expensive, and equal treatment, which requires a class of investors entitled to the same terms to receive the same payment, raises the price of assurance whichever kind of payment is used. Let h° be the smallest, over coalitions in $\mathcal{F}$ with at least two members, of the second-largest expense in the coalition, and let $\bar{m} = \max_{S \notin \mathcal{F}} |S|$ be the size of the largest failing coalition.

Proposition 49 (Assurance under equal treatment)**.** *Suppose no single investor can form the investment alone and all investors form a single class, each entitled to the same payment. A common escrowed payment implements formation if and only if it exceeds h°, and the infimum of its liability is $\bar{m}\, h^\circ \geq R^*$. A common own payment implements formation if and only if it exceeds h°, and it pays more than $n\, h^\circ$ in equilibrium, which no member of the class can finance without receiving it back.*

The common escrowed payment is the dominant assurance contract of Tabarrok (1998) with heterogeneous expenses. In the economy of Appendix B.3.3, where all four investors are indispensable and the expenses of standing ready are \$500, \$800, \$600 and \$1,000, the price of assurance rises from $R^* =$ \$1,900 to a common escrowed liability of $3 \times$ \$800 = \$2,400, against a common own outlay of more than \$3,200 that would have to come from outside the class. Equal treatment thus raises the price of assurance as Appendix B.9 shows it raises the price of retaining a pivotal investor, and it leaves the escrow's advantage, that nothing is paid in equilibrium, where it was. The proposition concerns public payments to investors who know their positions; Halac,

Lipnowski and Rappoport (2021) show that symmetric rewards can implement effort uniquely at lower cost when agents are uncertain of their rank in a sequence, so equal treatment is costly in the form studied here and not in every form.

B.9.3 Meeting the condition, and what it predicts

Condition $\Phi \geq C_k(z)$ can be met in four ways. The first raises Φ with funding from outside the entitled class, as when a sponsor forfeits founder shares or a manager or third party contributes beyond the rent; this pays the multiplier rather than avoiding it. The second shrinks $\mathcal{S}_k$ by placing the concession within an exception that the clause already contains, such as the exemption for terms tied to an investor's legal, regulatory or tax position described in Appendix B.18, at the cost of establishing qualification. The third obtains waivers, which Proposition 48 shows to be a coordination problem in its own right. The fourth, available under size-tiered clauses when commitments can be redesigned and allocations capped, resizes tickets so that fewer investors are entitled to the pivotal investor's terms.

The quantity that predicts failure is the pressure ratio $C_k(\Delta - s_k)/\Phi$, and its components are separately measurable: the pivotal investor's dollar shortfall, the commitments entitled to his terms, the differences between their existing terms and his, and the rent from which a concession can be funded. Commitment size enters through the entitled class rather than through the multiplier alone. When the shortfall is fixed in dollars, a smaller pivotal investor raises λ_k under universal and size-tiered clauses alike. When the shortfall is proportional to his commitment, $z = q_k\epsilon$, rate matching under universal equal treatment makes the cost $Q\epsilon$, where $Q = \sum_{j\in S} q_j$, independent of q_k, while a size-tiered clause makes it $\epsilon \sum_{j\in\mathcal{S}_k} q_j$, which is smaller for a larger member of the same program, because fewer investors are entitled to his terms. In a threshold program whose membership is minimal every member is pivotal, so the alternatives of members with high pressure ratios can govern formation. Excess separation should be most persistent where the entitled commitments are large relative to the rent available for concessions, where the entitled investors do not already hold better terms, and where entitlement is dispersed across many holders, so that waivers require coordination. A correlation between formation failures and the size of the pivotal investor is evidence for the mechanism only once these components are measured.

B.10 Threshold programs

In the binary specialization each investor subscribes or remains outside. Write S for the set of subscribers. The program forms if $|S| \geq \nu$, where $2 \leq \nu \leq n$. A member of a formed program obtains $u_i(S)$, inclusive of implementation costs and expenses of standing ready; a nonparticipant obtains o_i; a failed subscriber obtains $o_i - h_i$, with $h_i \geq 0$, an expense of standing ready that every subscriber bears whether or not the program forms. Assumptions 1 and 2 hold. The economic content lies in one assumption on formed payoffs, *inclusion monotonicity*:

$$S \subseteq T, \quad i \in S, \quad |S| \geq \nu \quad \Longrightarrow \quad u_i(T) \geq u_i(S). \tag{24}$$

An additional member weakly improves every existing member's net payoff. Program terms do not depend on outside values, any outsider may join a formed program, and there is neither rationing nor an admission veto. Scale economies of the usual kind satisfy (24): with fixed positive weights w_i and a fixed formation cost $F \geq 0$, the payoffs $u_i(S) = b_i - Fw_i / \sum_{j \in S} w_j - h_i$ do. A falling cost share is not sufficient when an entrant also dilutes a member's investment return; the assumption concerns net payoffs.

Proposition 50 (Completion for threshold programs)**.** *Under (24):*

(i) every individually acceptable profile is weakly Pareto-dominated by a pure equilibrium, so the game has completion;

(ii) the family of willing formed memberships, those with $u_i(S) \geq o_i$ for every member, is closed under union; if it is nonempty, its largest element $S^*(o)$ is a pure equilibrium that weakly Pareto-dominates every pure equilibrium, so $H(o) = W(S^*(o); o)$;

(iii) every acceptable formed membership that maximizes aggregate surplus among acceptable formed memberships is itself a pure equilibrium;

(iv) if $o' \geq o$ and both largest willing memberships exist, $S^*(o') \subseteq S^*(o)$.

The construction behind part (i) is admission. Start from an acceptable formed membership and admit any outsider who strictly gains by joining. Incumbents weakly gain by (24) and the entrant strictly gains, so acceptability is preserved, and after at most $n - |S|$ admissions no outsider gains by entering and no member gains by leaving. An acceptable failed profile gives every investor exactly his outside value and is dominated by the all-outside equilibrium. For part (iii), an outsider who strictly gained by joining a surplus-maximizing acceptable formed membership would create an acceptable formed membership with strictly larger surplus. Part (ii) is the willing-membership structure that underlies Moulin mechanisms for cross-monotonic cost sharing (Moulin, 1999; Moulin and Shenker, 2001; Juarez, 2013), and it yields a simple algorithm for the best equilibrium: begin with all investors, delete every member who is unwilling at the current membership, and repeat until the remaining set is willing or falls below the threshold. No member of $S^*(o)$ is ever deleted, because each obtains at least $u_i(S^*(o)) \geq o_i$ in any superset. Part (iv) holds because a higher outside value can remove willing memberships but cannot create them. Appendix A completes the proofs.

By Proposition 50, Theorem 32 applies to any outside-only improvement $o' = o + \Delta e_k$: the worst equilibrium rises by Δ, the selection gap weakly contracts, and the best equilibrium changes by an amount between $\Delta - G$ and Δ. The willing-membership structure thus locates the best equilibrium, whose advantage over the outside floor cannot grow when outside options improve.

An economy. Figure 2 computes the two gaps in a numerical five-investor economy. Gross formed values are $(12, 13, 11.4, 12.75, 12)$ thousand dollars, a fixed formation cost of 30 is shared equally, the program needs three members, and outside values $\theta(4, 8, 12, 18, 26)$ rise with a common technological improvement θ while program terms stay fixed. At $\theta = 0$ all five investors join

and the best equilibrium exceeds the floor by 31.15. Investor 5 drops out of the best membership when θ passes $6/26 \approx 0.231$, raising the others' cost shares, and investor 4 drops out when θ passes $7/24 \approx 0.292$, after which no three investors are willing. Panel A shows compression: the best equilibrium is constant until investor 5 leaves and falls discontinuously when he and then investor 4 leave, while the floor rises linearly, and the selection gap closes at $\theta \approx 0.292$, long before the program ceases to create surplus. Panel B shows where the surplus goes. The coordinated optimum stays at 31.15 until the outside floor overtakes it at $\theta \approx 0.458$, so for θ between 0.292 and 0.458 the entire value of coordination is distribution gap. The shaded region is the part a simple contract class recovers: named-member contracts with unanimous acceptance, balanced transfers of at most \$3,000 per member, and an administration charge of \$400 plus \$40 per member paid on formation. Those contracts support the attractive outcome in one equilibrium; they do not remove the all-outside equilibrium, which is the task of Section 3.

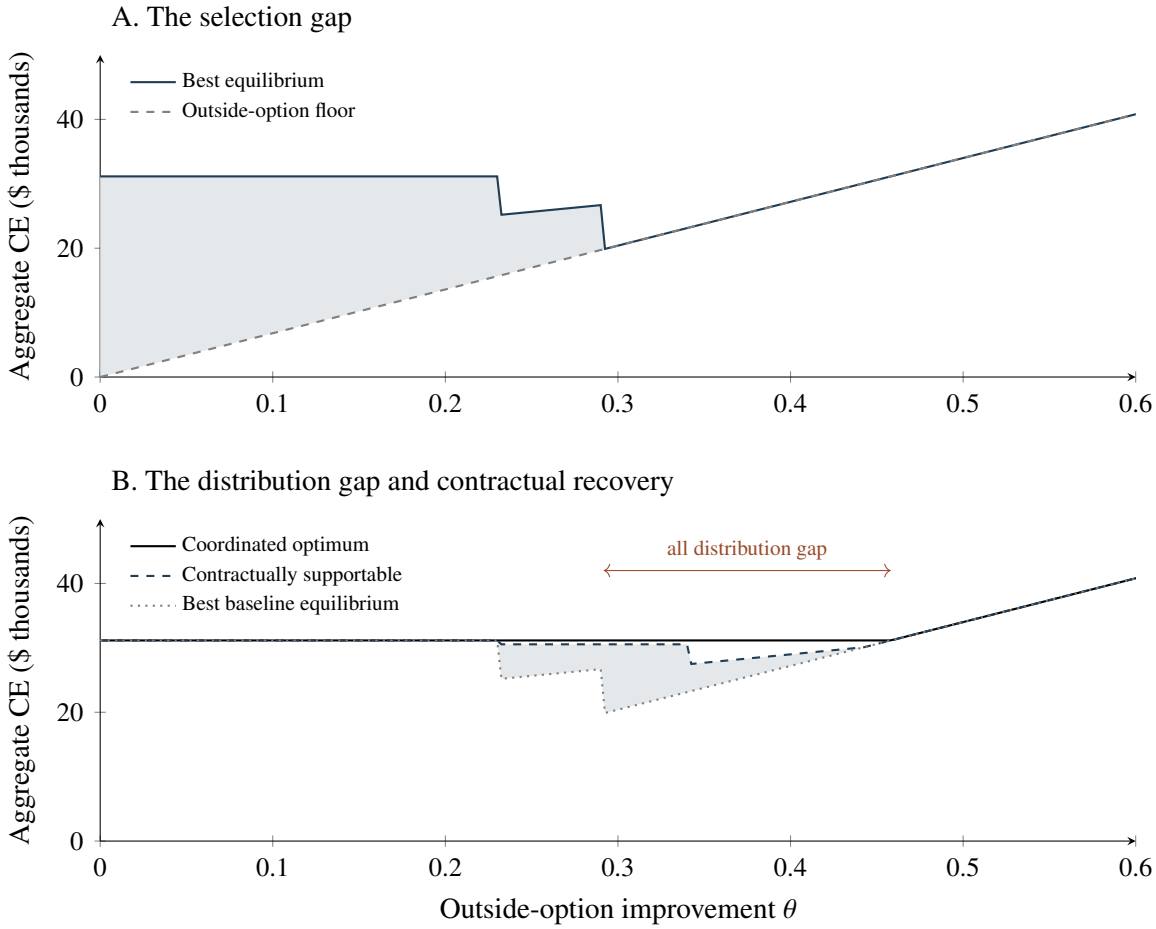


Figure 2: Compression and the two gaps. Five investors share a program that needs three members and costs 30; gross formed values are fixed while outside values rise to $\theta(4, 8, 12, 18, 26)$. Panel A shades the selection gap between the best pure equilibrium and the outside floor. Panel B compares the coordinated optimum with the best baseline equilibrium; because the economy has completion, the difference is the distribution gap, and the shaded part is what named-member contracts with balanced transfers of at most $3,000 per member recover after administration charges. All quantities are certainty-equivalent gains in thousands of dollars.

B.11 Divisible contributions

Completion also holds when indispensable investors contribute divisible capital to a program with a minimum scale, because an investor who moves toward his preferred contribution changes nobody else's payoff and preserves formation whenever the threshold is slack; the result requires neither a lattice of equilibria nor strategic complementarity (Theorem 52 in Appendix B.13). In a four-investor example with divisible capital (Appendix B.11.1), a fall from 10 to 7 in the per-unit cost of investing separately lowers best-equilibrium welfare from \$76,000 to \$22,000, although the shared program, which still requires the capital of the one investor whose separate investment has become attractive, would continue to produce \$76,000: he declines because proportional cost sharing would pay him \$19,000, less than the \$22,000 he now earns alone, and the selection term falls from \$60,000 to zero while the distribution term rises from zero to \$54,000. Each hypothesis of Theorem 32 is needed, and congestion, which arises whenever capacity is scarce, can reverse it (Proposition 54); compression should therefore appear where minimum scale or fixed formation costs bind, as in first closes, club transactions and programs that need a minimum committed amount, while where participation is congested the theory does not sign the effect.

B.11.1 A divisible-capital example

The lower bound of Theorem 32 is attained in an allocation problem with divisible capital. Four investors each hold two units of capital, one unit being \$100,000; values below are in thousands of dollars. Investor i may reserve $q_i \geq 0$ units for a common program and invest $x_i \geq 0$ units separately, with $q_i + x_i \leq 2$, holding the rest as cash. The program forms if $Q = \sum_i q_i \geq 7$, so it requires every investor. It returns 15 per unit, costs 40 in total, allocated in proportion to reserved capital, and charges a reservation cost of 0.5 per unit whether or not it forms. Separate investments return $(18, 2, 3, 4)$ per unit before an implementation cost of c per unit.

When the program forms, its net value per unit is $14.5 - 40/Q \leq 9.5$. At $c = 10$ every separate investment earns less per unit than that maximum, so the all-common profile gives each investor 19 and Pareto-dominates every other profile. Completion holds, the best equilibrium yields 76, and the floor is 16, the value of investor 1's separate investment. At $c = 7$ investor 1's separate investment earns 11 per unit, more than the program can pay, so any positive reservation is strictly dominated for him, and without his capital the others cannot reach seven units. The unique equilibrium yields 22, while the all-common profile, still feasible, yields 76; proportional cost sharing pays investor 1 only 19 of it. Contraction holds, because after the change investor 1 can obtain no more than his outside value in any profile and the others no more than 19, which the all-common profile delivered before. With $\Delta = 6$ and $G = 60$, the best equilibrium falls by $54 = G - \Delta$, and the entire value of coordination becomes distribution gap: $H - L$ falls from 60 to zero while $\Lambda = V - W^*$ rises from zero to 54, the game after the change having $W^* = H$ because no formed profile is acceptable to investor 1. A transfer of 3 to investor 1, out of the 57 that the other three gain, restores the all-common allocation only if it is tied to his reservation. A flat payment conditional on formation alone leads him to cut his reservation to one unit, which keeps

the program formed but lowers welfare to 72.5; a rebate of just over 1.5 per reserved unit restores the all-common allocation. Securing an investor's participation and fixing his quantity are different instruments, and Appendix B.9 shows that the per-unit form is also the one that equal-treatment clauses govern. Appendix A verifies these claims.

B.12 Completion and complementarity

Completion is distinct from strategic complementarity. Let investor i's return to subscribing, when the other subscribers are S, be the difference between subscribing and remaining outside. Before his subscription can meet the threshold that difference is $-h_i$; once his subscription would form the program it is $u_i(S \cup \{i\}) - o_i$.

Proposition 51 (Increasing differences)**.** *Under (24), the binary game has increasing differences if and only if*

$$u_i(T) \geq o_i - h_i \quad \textit{for every investor } i \textit{ and every membership } T \ni i \textit{ with } |T| = \nu. \tag{25}$$

If the smallest formed membership pays a member less than a failed subscription would, his return to subscribing falls at the threshold and the game lacks the cardinal increasing differences on which the classical results rest (Topkis, 1979; Milgrom and Roberts, 1990; Vives, 1990). When every expense of standing ready is positive, the game still satisfies the ordinal single-crossing condition of Milgrom and Shannon (1994), because the return to subscribing is negative until a subscription can form the program and nondecreasing thereafter, and $S^*(o)$ is then the greatest equilibrium of a game with ordinal complementarities when a willing formed membership exists, the all-outside profile being the greatest equilibrium otherwise. When expenses of standing ready are zero, even single crossing can fail. Completion, and therefore compression, holds in every case, so inclusion monotonicity implies completion without implying increasing differences, and the continuous model of Appendix B.13 shows that completion needs no lattice structure at all.

B.13 Continuous portfolios

B.13.1 Interior allocations and indispensable capital

An investor whose outside investment improves need not abandon a shared program; he may reduce his allocation to it. To distinguish adjusting a contribution from declining to contribute, let investor i choose a contribution $q_i \in [0, \bar{q}_i]$, and let the program form if $\sum_i q_i \geq K$. Capacities satisfy $\sum_i \bar{q}_i \geq K$ and $\sum_{j \neq i} \bar{q}_j < K$ for every i, so every investor is indispensable although his contribution is divisible. Capacities are net of protected liquidity and existing obligations. Fixed formation-cost shares are paid on success and do not vary with other investors' quantities; under pro-rata sharing, as in Appendix B.11.1, a contribution changes others' charges, and completion must be verified directly, as it is there.

A member's formed payoff is $o_i + f_i(q_i)$, where o_i is his optimized outside value. The incremental value f_i is continuous and strictly concave, with a unique maximizer $d_i \in (0, \bar{q}_i)$. Its value at zero may be negative, representing a fixed participation expense, and the results do not depend on its sign. A reservation costs $\rho_i q_i$, with $\rho_i > 0$, whether or not the program forms; the cost is included in f_i, and a failed reservation returns capital and yields $o_i - \rho_i q_i$. There is no congestion, and Assumptions 1 and 2 hold because any investor's refusal destroys formation.

These assumptions accommodate genuine portfolio mixing. One representation is

$$f_i(q; t) = P_i(q) + \chi_i(B_i - q; t) - \chi_i(B_i; t) - F_i - \rho_i q, \tag{26}$$

where P_i values the shared position, $\chi_i(w; t)$ values an outside investment of retained capital w that the investor holds alongside it, B_i is investable capital after a protected liquid reserve, F_i is his fixed expense share and t indexes outside technology. Concave components give a concave f_i, and one strictly concave component gives strict concavity.

Theorem 52 (Continuous formation)**.** *In the indispensable-cohort game:*

(i) all-zero is the only failed pure equilibrium, and a formed profile q is a pure equilibrium if and only if either $\sum_i q_i > K$ and $q_i = d_i$ with $f_i(d_i) \geq 0$ for every i, or $\sum_i q_i = K$ and $q_i \geq d_i$ with $f_i(q_i) \geq 0$ for every i;

(ii) a formed equilibrium exists if and only if every peak $f_i(d_i)$ is nonnegative and $\sum_i R_i \geq K$, where, for a nonnegative peak, $R_i = \max\{q \in [d_i, \bar{q}_i] : f_i(q) \geq 0\}$ is investor i's largest acceptable contribution;

(iii) the game has completion;

(iv) when an acceptable formed allocation exists, the one that maximizes $\sum_i f_i(q_i)$ is unique and is a pure equilibrium, and

$$G = \max\left\{0, \max_{\substack{0 \leq q_i \leq \bar{q}_i,\ \sum_i q_i \geq K \\ f_i(q_i) \geq 0\ \forall i}} \sum_i f_i(q_i)\right\}, \tag{27}$$

where the inner maximum is omitted when no acceptable formed allocation exists.

Two forces determine the equilibria: in a failed profile a positive reservation can be canceled to avoid its cost, and in a formed profile with slack an investor away from his preferred contribution can move toward it without destroying formation, while exactly at the threshold an investor below d_i gains by contributing more and any reduction destroys formation, so that the threshold can sustain a contribution above an investor's private optimum, provided the contribution remains individually acceptable.

Completion follows by a construction different from admission. From an acceptable formed allocation, raise every contribution that lies below its owner's peak to that peak. If the peaks together meet the threshold, the vector d is an equilibrium that weakly improves everyone.

Otherwise lower the contributions that lie above their peaks until the threshold binds. Each move brings a contribution toward its owner's optimum, and the result satisfies the threshold conditions in (i). Part (iv) then follows from Theorem 31: the acceptable domain is convex, the objective strictly concave, and the dominating equilibrium of the unique maximizer can only be the maximizer itself.

In this model the best individually acceptable coordinated allocation is therefore itself a Nash equilibrium, and the mechanism behind completion is visible: because payoffs are separable and expense shares fixed, an investor who moves toward his preferred contribution changes no other investor's formed payoff, and he preserves formation whenever the threshold is slack.

The equilibrium set need not have a greatest element. With two investors, capacities of one, $K = 1.6$ and $f_i(q) = 1 - 5(q - 0.6)^2$, every profile on the threshold with both contributions in $[0.6, 1]$ is an equilibrium, including $(0.6, 1)$ and $(1, 0.6)$. Their componentwise maximum $(1, 1)$ is not, because either investor can reduce his contribution toward 0.6 while the program remains formed. Monotone membership in the binary game has no counterpart in a lattice of contributions, and completion does not need one.

B.13.2 Two kinds of technological change

An improvement available only to an investor who stays out entirely raises o_i by δ_i and lowers every positive-participation value f_i by δ_i. An improvement that the investor can mix with the program enters both χ_i terms in (26), and the two experiments differ. If χ_i is increasing in t and has increasing differences in retained capital w and technology t, then the outside value rises and the all-outside position, which retains the most capital, benefits at least as much as any partial position, so $f_i(q; t') \le f_i(q; t)$ pointwise. In a differentiable interior model, $\partial^2 \chi_i / \partial w \, \partial t \ge 0$ gives $\partial^2 f_i / \partial q \, \partial t \le 0$ and a preferred contribution d_i that falls with technology. This is a standard monotone comparative-statics argument (Milgrom and Shannon, 1994); concavity alone does not supply the sign.

Corollary 53 (Continuous compression)**.** *Hold the cohort, capacities, threshold and reservation costs fixed. If $o_i' \ge o_i$ and $f_i' \le f_i$ pointwise, then (13) holds with $\Delta = \sum_i (o_i' - o_i)$.*

The corollary is Theorem 32 applied to Theorem 52, and it needs no greatest equilibrium. Its hypothesis also marks its boundary. Declining per-unit charges, changing capacities or an outside investment that hedges the shared position do not by themselves deliver the pointwise order. A hedge that is especially valuable alongside the shared investment can raise f_i, and the selection gap can then widen even as the outside value rises.

B.13.3 A cliff with interior portfolios

Partial adjustment need not prevent collapse, as a four-investor economy shows. Each investor holds two units of risky capital, one unit being \$100,000, and a separate protected liquid reserve. Shared capacities are $(1, 1.1, 0.9, 1.2)$ and the program needs 3.5 units. In thousands of dollars,

$$f_i(q_i;\theta) = \kappa_i(\theta)\, q_i - 10q_i^2 - F_i, \qquad \kappa(\theta) = (17-\theta,\ 18.4,\ 15.6,\ 20), \qquad F = (6,2,2,2). \quad (28)$$

These values arise from (26) with $B_i = 2$, a shared position worth $P_i(q) = \psi_i q - 5q^2$, an outside investment worth $\chi_i(w;\theta) = \mu_i w - 5w^2 + \theta_i w$ with $\mu_i \geq 20$, $\psi_i - \mu_i = \kappa_i(0) - 19.8$, a reservation cost of $0.2q_i$, and $\theta_1 = \theta$ while $\theta_j = 0$ for the others. Only investor 1's outside technology improves. Because $\mu_1 \geq 20$, his outside position invests all his capital, his outside value rises by 2θ, and his incremental values fall pointwise.

At $\theta = 0$ the preferred contributions are $(0.85, 0.92, 0.78, 1)$, totaling 3.55. Every choice is interior, every portfolio retains the outside investment and the liquid reserve, and the threshold is slack. As θ rises investor 1 contributes less. Beyond $\theta = 1$ the threshold binds, and every investor contributes slightly more than his private optimum so that the program stays formed; at $\theta = 1.5$ the best equilibrium is $(0.78125, 0.92625, 0.78625, 1.00625)$. Until $\theta \approx 1.5075$ the coordinated optimum is individually acceptable and is therefore, by part (iv) of Theorem 52, the best equilibrium itself. Beyond that point investor 1's acceptability binds: the best equilibrium holds his contribution at its largest acceptable level and lies below the coordinated optimum, though by at most about \$0.54, so the binding constraint costs almost nothing before the cliff. At $\theta = 17 - \sqrt{240} \approx 1.5081$ investor 1's best participation value reaches zero, and for every larger θ no formed equilibrium remains. Immediately beyond that point the coordinated allocation still produces more than \$18,500 of surplus above the outside floor, all of it distribution gap. Figure 3 shows the smooth adjustment and the cliff.

A second economy isolates a shortfall of acceptable capital with no negative peak. In thousands of dollars, let three investors have capacities 2, $K = 4.7$, and $f_i(q) = \bar{f}_i - (q - d_i)^2$ with $d = (0.5, 1.5, 1.5)$ and peaks $\bar{f} = (0.2, 2, 2)$. The largest acceptable contributions sum to 4.9472. An outside-only improvement of 0.17 for investor 1 lowers his peak to 0.03 and the sum to 4.6732, below the threshold, although every peak remains positive. The feasible allocation $(0.7, 2, 2)$ still yields increments $(-0.01, 1.75, 1.75)$, a total of 3.49. Compensating a \$10 loss would preserve \$3,490 of surplus, and neither a better optimizer nor a better equilibrium-selection device supplies that compensation.

B.14 The boundary of compression

Theorems 31 and 32 rest on four hypotheses: completion, contraction, independent outside options and the absence of failure rewards. Each is needed for at least one of them.

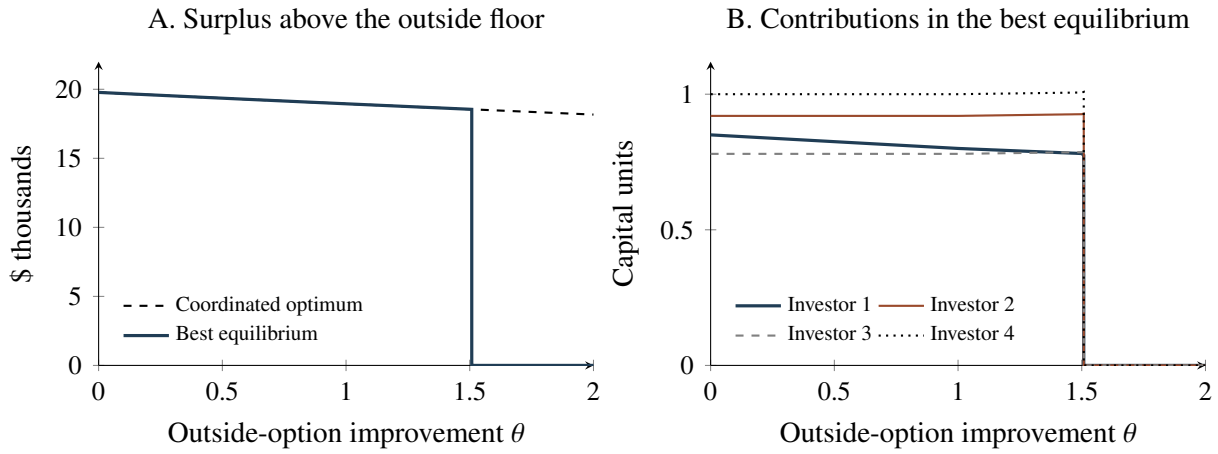


Figure 3: Partial adjustment and a cliff. Four investors with interior portfolios share a program that needs 3.5 capital units, and only investor 1's outside technology improves with θ. In Panel A the best equilibrium coincides with the coordinated optimum, as Theorem 52 implies, until investor 1's acceptability binds at $\theta \approx 1.5075$; the two then differ by at most about \$0.54 until his best participation value reaches zero at $\theta = 17 - \sqrt{240} \approx 1.508$, beyond which formation fails although the coordinated allocation still creates more than \$18,500 of surplus. Panel B shows contributions adjusting smoothly before the cliff. Thousands of dollars; one capital unit is \$100,000.

Proposition 54 (Each hypothesis is needed)**.** *In each game of Table 6 the hypothesis named in its row fails and the others relevant to the result hold, and the result named there fails: the bounds (13) in the first two rows, and the floor of Lemma 2 in the last two.*

Congestion, the first failure, arises whenever capacity is scarce: an oversubscribed co-investment, a strategy with limited capacity, a credit facility with concentration limits. An entrant then gains at the incumbents' expense, completing an acceptable membership need not dominate it, and an improved outside option can deter harmful entry instead of destroying formation. In the first game of Table 6, any pair is acceptable, but the excluded investor enters and reduces its welfare from 20 to 3; once his outside value exceeds what entry would pay him, the valuable pair becomes an equilibrium and everyone gains. Congestion does not reverse compression in general, however. If investor 3's outside value rises only to 0.5, he still joins, the full program remains the best equilibrium, and the selection gap compresses from 3 to 2.5. Under congestion the sign of the effect is not determined; Proposition 55 bounds how far the range can widen.

Contraction fails when the change that improves outside options also changes the program's terms, so the experiment must hold terms fixed or measure their change. Independence fails when outside alternatives compete for the same scarce asset, in which case the sum of separate optima is not jointly attainable and is not an equilibrium floor. In the third game the program, acceptable against the payoffs $(5, 5)$ of staying out together, is not an equilibrium, because each investor would leave to take the whole asset; comparing participation with the jointly rationed outside payoff misstates the incentive to refuse. The last row describes the mechanism of Section 3: a payment upon failure moves the floor, and such payments are how assurance removes bad equilibria.

Table 6: The hypotheses of compression

Hypothesis	Game	Result that fails
Completion	Three investors, threshold two, outside values zero. Each member of a pair receives 10; each member of the full program receives 1.	Raising investor 3's outside value to 1.5 widens the selection gap from 3 to 20 and raises the best equilibrium by 18.5.
Contraction	Two investors, both required, outside values zero, formed payoffs (10, 10). Investor 1's outside value rises to 1 while formed payoffs rise to (12, 12).	The selection gap rises from 20 to 23.
Independent outside options	Two investors, both required. An outside asset worth 10 is split equally if both stay out and taken whole by a lone outsider, so each investor's outside value is 10. The program pays 6 to each.	Staying out together is the only equilibrium, and it yields 10 rather than $\sum_i o_i = 20$, so Lemma 2 fails.
No failure reward	Two investors, both required, outside values zero, formed payoffs $(-2, -2)$. A lone subscriber receives 1.	Staying out together is not an equilibrium, and the worst equilibrium yields 1, so Lemma 2 fails.

The improved alternative may itself be a shared program. Four investors choose between program X, which needs three members and pays each member 10 less an equal share of a cost of 3, and program Y, which any investor can form alone and which pays (14, 14, 1, 1) less an equal share of its cost. When the cost of program Y falls from 12 to 2, best-equilibrium welfare falls from 37 to 28, while coordinated welfare rises from 37 to 39. The first two investors move to program Y, the other two follow to share its cost, and program X can no longer form. Because the payoff of program Y depends on how many choose it, the outside alternative is not independent, and the two programs must be analyzed jointly (Internet Appendix IV).

Entry is outside the indispensable-cohort model and can change its conclusions. An entrant may replace an investor who declines, lower cost shares or supply the acceptable capital that a shortfall requires, and the pivotal condition for a fixed membership must then be recomputed. Appendix B.17 gives an example in which falling costs draw a new investor into a program.

The boundary delivers the theory's first empirical prediction, stated in Section 6; where participation is congested an improved outside option can compress the range or, by deterring harmful entry, widen it, and asset-class labels do not decide the case, because the same manager's first fund may be limited by minimum scale and its later co-investments by capacity, and the relevant observation is how one more compatible participant changes the incumbents' net terms.

Completion is stronger than the scalar result of Theorem 32 requires. The proof uses only $H(o) = W^*(o)$, and the following bound shows how compression degrades when that equality fails.

Proposition 55 (Approximate compression)**.** *Suppose Assumptions 1 and 2 hold at o and $o' = o+\delta$ and the change contracts acceptable payoffs, but the game at o need not have completion. Let $U(o) = W^*(o) - H(o) \geq 0$ be the instability term of (6). Then*

$$G(o') \leq G(o) + U(o), \qquad H(o') - H(o) \leq \Delta + U(o), \tag{29}$$

and both bounds are attained.

The proof repeats the chain in Theorem 32, stopping at $\sup_{\mathcal{A}(o)} \sum_i g_i(a; o) = W^*(o) - L(o) = G(o) + U(o)$. Theorem 32 is the case $U(o) = 0$. The congestion game of this appendix attains both bounds: there $U(o) = 17$, the selection gap rises from 3 to exactly $3 + 17$, and the best equilibrium rises by exactly $\Delta + 17$. The range can widen only by the surplus that acceptable but unsupported allocations held in reserve before the change. Completion is needed only before the change, and the game after it need not have the property.

B.15 Excess separation

For a given allocation, the compression of Section 6 takes a more specific form. Fix an acceptable formed profile a with slacks $s_i = g_i(a; o) \geq 0$, and let the outside value of an investor k who participates in a rise by Δ through an outside-only improvement.

Lemma 56 (Excess separation)**.** *Let k be a participating investor, $a_k \neq 0$. The profile a remains acceptable if and only if $\Delta \leq s_k$, and it continues to create surplus above the new floor if and only if $\Delta < \sum_i s_i$. In the interval*

$$s_k < \Delta < \sum_i s_i, \tag{30}$$

a balanced transfer from the other participants to investor k restores acceptability if and only if it is at least $\Delta - s_k$ and at most $\sum_{i \neq k} s_i$. Under a pointwise decline on all profiles the total value of coordination $V - L$ weakly falls, while the distribution gap Λ can rise.

The first statements are arithmetic: investor k's slack falls to $s_k - \Delta$ and aggregate surplus to $\sum_i s_i - \Delta$. Under a pointwise decline every profile's aggregate surplus weakly falls, so its supremum $V - L$ does too, while in the example that attains the lower bound of Theorem 32 the distribution gap rises from zero to 9.9. In the interval (30) an improved outside option converts selection surplus into distribution gap. In that example the program still yields 10.5 after the improvement, while the all-outside allocation yields only 0.6, and a transfer of at least 0.1 from the second investor to the first restores formation; because both investors are required, the transferred program is an equilibrium, since either investor's refusal would destroy it. The resources and the investment's gross value survive the improvement, and what fails is the distribution of surplus. Whether it can be repaired depends on whether a transfer of the required size is admissible under each mandate, collectible when due and enforceable. Figure 4 draws the interval in the space of incremental payoffs.

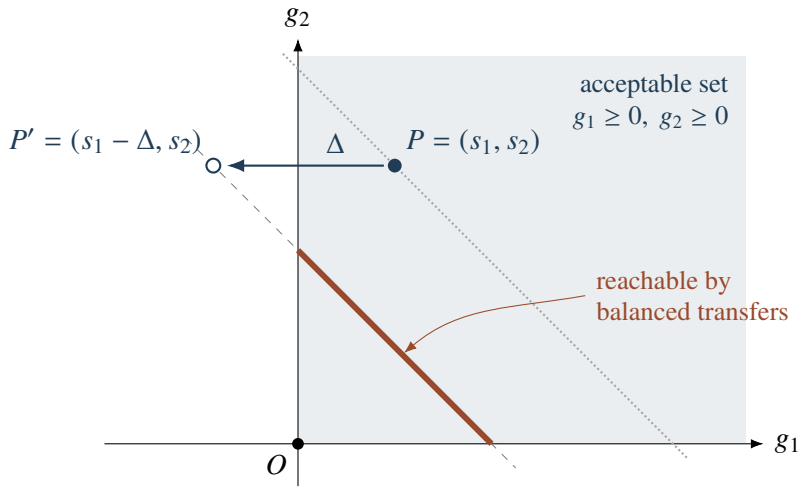


Figure 4: Excess separation in incremental payoffs. The shaded orthant contains the acceptable incremental payoff vectors; O is the all-outside equilibrium. A formed allocation P gives slacks (s_1, s_2). An outside-only improvement of Δ in investor 1's outside option moves it to P', outside the acceptable set although its aggregate surplus $s_1 + s_2 - \Delta$ remains positive. The dotted and dashed lines hold aggregate surplus at its level before and after the change. Balanced transfers move P' along the dashed line, and the thick segment is the set of acceptable allocations they reach.

B.16 Technology and collective search

Technology acts on two different costs, and Definition 1 tells them apart. The first is the cost of an own term: interpreting a mandate, maintaining an attributable record of an account's holdings and outcomes, implementing a portfolio for one investor and paying a manager on that account's record. Lowering it raises outside values and completes records, which by Corollary 30 makes delegation feasible account by account and leaves to joint terms the roles of lowering its cost and of committing accounts together. The second is the cost of a joint term: finding the investors whose mandates admit a program, establishing that each is eligible and authorized to commit, holding their commitments in escrow, computing the liability an escrowed schedule creates in every subscription profile, which with unequal tickets is a knapsack problem (Komkov, 2026b), and paying a manager on a joint record. Lowering the second enlarges what joint terms can achieve, so that the first cost acts on the floor of the equilibrium range and the second on its ceiling. Whether a technology that lowers both makes an investor less willing depends on incidence, the saving that reaches him outside the program net of the saving that reaches him inside it, compared with the slack he already has, which Appendix B.17 formalizes; a common percentage improvement does not produce common dollar savings, so a uniform improvement can have a concentrated effect on participation.

Collective search. Technology also changes who can take part. Artificial intelligence lowers the cost of collective search, the joint-term tasks of finding eligible and authorized investors and

assembling their commitments on common terms listed above. These tasks are performed once for a program rather than once for each investor's separate portfolio, and their cost determines which investors a program can reach. Let N be the investors who could in principle take part and $P \subseteq N$ the pool that a program can reach at a given cost of search. An investor outside the pool can only remain outside and obtains his outside value. In the threshold model of Appendix B.10, write $L(o; P)$, $H(o; P)$, $G(o; P)$ and $W^*(o; P)$ for the extreme equilibrium welfare levels, the selection gap and the best acceptable welfare of the game played by the pool, with welfare summed over all of N.

Proposition 57 (Collective search)**.** *Under inclusion monotonicity (24), let $P \subseteq P' \subseteq N$. Then:*

(i) the floor does not move: $L(o; P') = L(o; P) = \sum_{i \in N} o_i$;

(ii) the best equilibrium of the larger pool weakly Pareto-dominates the best equilibrium of the smaller one, and if the smaller pool contains a willing formed membership, its largest willing membership is contained in the larger pool's;

(iii) the best equilibrium and the selection gap rise by the gain in the best acceptable surplus:

$$H(o; P') - H(o; P) = G(o; P') - G(o; P) = W^*(o; P') - W^*(o; P) \geq 0. \tag{31}$$

Wider search is the mirror image of a better outside option. A better outside option raises the floor and, under completion, compresses the range above it; a larger pool leaves the floor where it is and weakly raises the ceiling, weakly widening the selection gap. The first shrinks the family of willing memberships, by Proposition 50(iv), and the second enlarges the set of investors from which willing memberships are drawn. Theorem 31(iii) explains why search is the margin on which coordination at fixed terms can raise the best attainable outcome: computing the best allocation for a given pool cannot improve on the best equilibrium, but finding the investors who belong in it can. Because a larger pool weakly raises the selection gap, it also raises what selection instruments can recover, and it weakly lowers the least reserve that assurance requires, since it adds forming coalitions without removing any (Proposition 43).

Return to the five-investor economy of Appendix B.10, in which outside values rise to $\theta(4, 8, 12, 18, 26)$ under fixed terms and the program fails once θ exceeds $7/24 \approx 0.292$. Suppose that cheaper search reaches a sixth investor identical to investor 1, with gross value 12 and outside value 4θ. The program then forms with all six investors until $\theta = 7/26 \approx 0.269$ and with five until $\theta = 0.375$. For θ between 0.292 and 0.375 the original pool has no formed equilibrium, while the enlarged pool supports one in which investors 1, 2, 3, 4 and 6 all gain; at $\theta = 0.3$ its best equilibrium exceeds the unchanged floor by 17.35. Reaching one investor whose alternative is no better than investor 1's offsets an improvement in outside options that had destroyed the program. Figure 5 draws the two selection gaps.

The result needs inclusion monotonicity. Under congestion an added investor can harm incumbents, and a larger pool can lower the best equilibrium: in the congestion game of Table 6 in Appendix B.14, a pool of investors 1 and 2 supports their pair with welfare 20, while adding

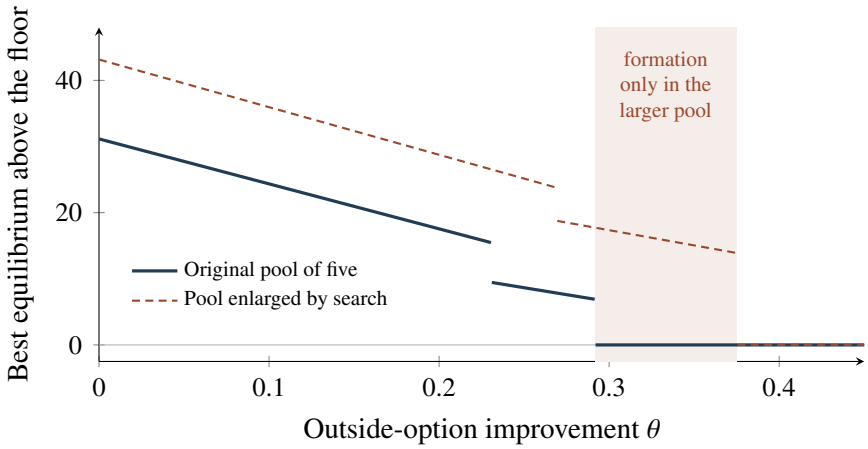


Figure 5: Collective search and the selection gap. The five-investor economy of Figure 2, with outside values $\theta(4, 8, 12, 18, 26)$ and fixed terms, is compared with the same economy after search reaches a sixth investor with gross value 12 and outside value 4θ. The floor is the same in both. The larger pool raises the best equilibrium at every θ below 0.375 and keeps the program formed for θ between $7/24$ and 0.375, where the original pool has already failed; beyond 0.375 neither pool forms the program. Breaks mark changes in membership. Thousands of dollars.

investor 3 to the pool leaves a best equilibrium of 3. A search technology can also raise outside values, by helping each investor find separate investments, and then its effect on formation combines both forces of this section. The net effect of a technology that does both depends on their relative incidence, as the effect of cost reductions depends on incidence across separate and shared implementation (Appendix B.17), and part (ii) of Proposition 43 adds a further channel, through which search that leads many investors to the same recommendation raises the price of the discretion a book needs. Appendix B.18 reports a companion experiment that measures one component of the incidence comparison for AI-assisted investment administration.

B.17 Incidence

Technology rarely improves only the outside option. Artificial intelligence, for example, lowers the cost of interpreting a mandate, reconciling an amendment and maintaining an attributable record of holdings, and those tasks are performed both in a separate account and inside a shared program. Write a member's formed payoff as $b_i - \phi_i(S)$, where ϕ_i includes his real implementation burden and his allocated share of common costs. Holding the membership and gross values fixed, let outside values change by Δo_i and burdens move from ϕ_i to ϕ_i'. Individual willingness survives exactly when

$$\Delta o_i \le s_i + \phi_i(S) - \phi_i'(S) \qquad \text{for every } i \in S. \tag{32}$$

If technology changes outside values only through implementation costs, with σ_i^{sep} the reduction in investor i's cost of acting separately and σ_i^{sh} the reduction in his cost inside the program, the condition becomes

$$\sigma_i^{\text{sep}} - \sigma_i^{\text{sh}} \le s_i. \tag{33}$$

The relevant quantity is incidence: the saving that reaches an investor outside, net of the saving that reaches him inside, compared with the slack he already has. A common percentage improvement does not produce common dollar savings. If initial separate and allocated shared costs are c_i^{sep} and c_i^{sh} and both fall by the fraction θ, willingness requires

$$\theta\left(c_i^{\text{sep}} - c_i^{\text{sh}}\right) \le s_i \qquad \text{for every member } i, \tag{34}$$

and among members whose separate costs exceed their shared costs, the first to decline is the member with the smallest ratio $s_i/(c_i^{\text{sep}} - c_i^{\text{sh}})$. A uniform technological improvement can therefore have a concentrated effect on participation. The investor with the most distinctive mandate is pivotal only if his cost incidence is large relative to his slack; complexity alone does not determine who declines.

The ratio describes a fixed membership and is incomplete under open entry, because falling costs can also draw in a newcomer whose entry lowers everyone's charges. With gross values $(12, 12, 12, 5)$, outside values $(1, 1, 1, 0)$, a threshold of three and a common expense of $24(1 - \theta)$, the fourth investor stays out until $\theta = 1/6$, when an equal share of the expense falls to 5 and he joins, enlarging the best membership from three to four and lowering each incumbent's charge.

B.18 The technology economy and a measured instance

Figure 6 returns to the five-investor economy of Appendix B.10 and lets technology lower the common expense from 30 to $30(1 - \theta)$ while outside values rise to $\theta(4, 8, 12, 18, 26)$. Both routes improve, and the shared program's cost falls by the same percentage for everyone. The best membership nevertheless shrinks as θ rises, in the order of the investors' outside gains. Investor 5, whose outside route gains most, no longer joins once θ exceeds 0.3; the remaining members' cost shares rise, investor 4 drops out at $\theta = 0.5$ and investor 3 at $\theta = 0.7$, beyond which the program cannot form although its expense has fallen by 70 percent. At each of these points the best equilibrium drops below the coordinated optimum, and the difference is distribution gap. Each value of θ describes a separate formation decision, not a later stage of one program; the calculation enumerates every pure equilibrium at each point and permits entry, and along this path the best membership only shrinks.

A companion experiment measures one component of the incidence comparison for AI-assisted investment administration (Komkov, 2026a). It interprets 64 simulated cases, each with four investors observed at four dates, once separately for each investor and once jointly with shared evidence, holding information and required work fixed, and it compares retrieval alone with persistent records and validation. Table 7 reports the magnitudes. Shared processing costs less in

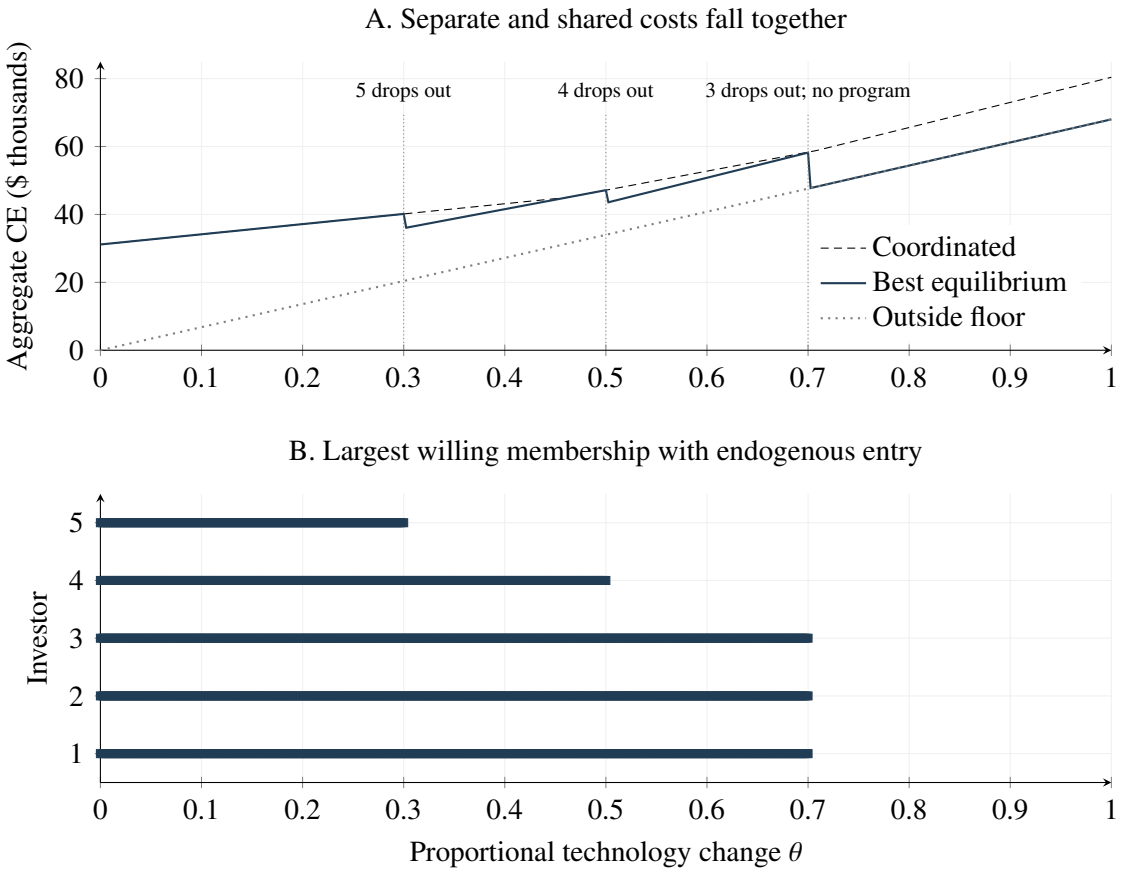


Figure 6: Cost incidence and membership. The common expense falls from 30 to $30(1-\theta)$ while outside values rise to $\theta(4, 8, 12, 18, 26)$. Panel A enumerates every pure equilibrium at each point and reports the coordinated optimum, the best equilibrium and the outside floor. Panel B marks membership in a largest best formed equilibrium, retaining ties. Each value of θ is a separate formation decision. Entry is permitted; along this path the best membership only shrinks. Thousands of dollars.

level, while the improvement saves more in separate processing, because it removes retrieval that separate processing repeats.

Table 7: Measured cost incidence in the companion experiment

Model-usage cost per case ($)	Separate	Shared
Retrieval only	0.3311	0.1457
Records and validation	0.2970	0.1398
Saving	0.0341	0.0059
Separate-minus-shared saving: $0.028 per case (95% interval $0.017 to $0.039), $0.007 per investor		

Each case contains four investors observed at four dates. Source: Komkov (2026a).

The sign holds for model-usage costs: the separate-minus-shared saving is positive in aggregate and for 164 of 256 investors under equal allocation of shared costs, although individual signs differ. Its size, under a cent per investor, is far too small to move a participation decision, and the costs that could, chiefly human review, enter the experiment only through assumed prices, under which the incidence rises to between $0.26 and $1.20 per case and becomes statistically indistinguishable from zero. The experiment therefore establishes that a common technological improvement can save more in separate processing than in shared processing, which is what (33) needs in order to matter, and leaves the magnitude in a real office to measurement. Which effect dominates for a pivotal investor, and whether a provider passes the saving through or retains it as margin, determines whether the technology strengthens shared investments or dissolves them.

Technology also acts on the repair. Where an equal-treatment clause exempts concessions tied to an investor's legal, regulatory or tax position, a concession can be confined to the entitled investors who share the qualifying circumstances, at a cost κ^{q} of establishing qualification. Under rate matching, securing the investor then requires $\min\{\lambda_k z,\ \lambda_k^{\mathrm{q}} z + \kappa^{\mathrm{q}}\} \le \Phi$, where λ_k^{q} is the multiplier computed over the qualifying class. Persistent, attributable records of the kind that cheapen separate implementation can lower κ^{q}, and they can lower the review cost of the waivers of Appendix B.9. They do not change who holds an entitlement, however, and they can act in the other direction, by making preferential treatment easier to detect and entitlements easier to enforce. Where the first effect operates, a technology raises outside values, which creates excess separation, while lowering the cost of confining a concession, which makes excess separation cheaper to repair. Confinement is used when $(\lambda_k - \lambda_k^{\mathrm{q}})z > \kappa^{\mathrm{q}}$, and within that regime a rise dz in the shortfall leaves the concession affordable if the cost of qualification falls by at least $\lambda_k^{\mathrm{q}}\, dz$. Which effect dominates in practice is a hypothesis for measurement rather than a consequence of the model.

B.19 The cost of commitment support: a sensitivity analysis

Of the four quantities that Section 7.2 keeps apart, this subsection computes the first two under the paper's benchmark, the capacity a promisor must hold (R^*, or $\hat{R}_\ell$ under a staged escrow) and

the carry of holding it over the closing window $\bar{\tau}$; its input, the expense of standing ready, is an assumption rather than an estimate. The holding cost is not the yield on the reserve, which a reserve held in Treasury bills earns, but the spread between that yield and the return on the reserve's best alternative use. For a bank that supplies the capacity as an undrawn line of credit, the commitment fee on such lines, which Berg, Saunders and Steffen (2016) show to be a principal component of the cost of committed credit and which is of the order of tens of basis points a year, is a proxy, although the failure payment differs from a drawdown option in its recourse, collateral and draw conditions and would be priced on its own terms; for a sponsor that holds the reserve itself it is the spread between its own cost of capital and the bill yield, about 8 to 11 percent for a sponsor whose capital costs 12 to 15 percent, and ι denotes the annual cost in either case. The expense of standing ready is the cost an investor bears between committing and closing that is lost if the investment fails: the negotiation of subscription documents and side letters, internal approvals, and the reservation of capital or of an allocation that could have been committed elsewhere. Due diligence performed before the decision to commit is sunk at that decision and is not part of it.

Because the expense of standing ready is not observed, the calculation is conditional on it, and the comparisons below are statements about the values of that expense under which they hold. Let a fund of size nq be raised from n investors of equal size q whose expenses of standing ready are a fraction θ of their commitments, so that $h = \theta q$, and let formation require m of them, so that $R^{*} = (m-1)\theta q$. The holding cost of the capacity over a window $\bar{\tau}$ is $\iota\bar{\tau}(m-1)\theta q$, which as a share of the fund is at most $\iota\bar{\tau}\theta$. Organizational expenses at a median cap of 20 to 25 basis points exceed that share unless θ exceeds $0.0020/(\iota\bar{\tau})$: with a capacity cost of 50 basis points a year and a closing window of a year, unless each investor's expense of standing ready exceeds 40 percent of his commitment, and with a capacity cost of 10 percent, a sponsor's cost of holding the reserve itself, unless it exceeds 2 percent. The present value of management fees, about \$10 to \$15 per \$100 committed (Metrick and Yasuda, 2010), exceeds the holding cost unless θ exceeds $0.10/(\iota\bar{\tau})$, which no positive expense of standing ready smaller than the commitment itself satisfies at either capacity cost. Table 8 sets out the comparison for a fund of \$300 million raised from 30 investors, with expenses of standing ready of one percent of each commitment, and with formation requiring all 30 or the ten whose commitments a first closing at a third of the target would need.

Under the known-willingness benchmark, the carry of the capacity needed to eliminate strategic nonformation can therefore be small relative to observed fund charges: when expenses of standing ready are a full percent of each commitment and every investor is required, the capacity is about one percent of the fund, and its carry runs from half a basis point, when a bank supplies it, to about ten, when a sponsor holds it. That is a statement about one narrow function and motivates measurement; it does not establish what the remaining charges pay for. The manager's own commitment is between about 1.8 and 2.6 times the capacity when all 30 investors are required, and between about 6 and 8.5 times when ten are, but it is committed on the path on which the fund forms, and pledging part of it to investors' expenses upon a failed closing would require a financing and contractual arrangement that the comparison of amounts does not establish. By

Table 8: Contingent capacity and what investors pay: a $300 million fund with 30 investors

	$ thousands	Basis points
Expense of standing ready, $h = 1\% \times \$10$ million	100	3.3
Least liability, all 30 required, $R^* = 29h$	2,900	97
Least liability, 10 required, $R^* = 9h$	900	30
All 30 required, three public rounds, $\hat{R}_3 = 9h$	900	30
Holding cost of $29h$, a year at 50 basis points	14.5	0.5
Holding cost of $29h$, a year at 10 percent	290	9.7
Organizational expenses at median cap	600–750	20–25
Manager's commitment (median, venture to buyout)	5,100–7,650	170–255
Present value of management fees	31,000–44,000	1,030–1,460

The expense of standing ready is an assumption. Organizational expenses: median caps reported in Institutional Limited Partners Association (2026). Manager's commitment: Carta (2025). Management fees: median present value per $100 committed, buyout to venture, from Metrick and Yasuda (2010). The capacity cost of 50 basis points is of the order of commitment fees on undrawn credit lines (Berg, Saunders and Steffen, 2016); 10 percent approximates the spread between a sponsor's cost of capital and the Treasury-bill yield, the cost of holding the reserve itself.

Proposition 8, a sponsor or platform that backs many vehicles with disjoint investors needs only the largest of their requirements, so that shared contingent support can serve separately owned accounts and vehicles. Corollary 6 shows that a promisor with capacity marginally above R^* could collect nearly the whole surplus of formation under the model's contracting assumptions; whether any observed charge is such a rent is an empirical question about competition among promisors and about the other services charges pay for, and the model does not answer it.

B.20 One transaction: FirstMark Horizon and Starry

The 2022 combination of FirstMark Horizon Acquisition Corporation with Starry separates participation rights, terms, permission and cash in a single documented transaction (Figure 7). The October 2021 merger agreement made closing conditional on at least $300 million of defined closing cash, a formation threshold in the sense of the model. The definition counted trust cash net of redemptions and the proceeds of committed equity, and it excluded $150 million of convertible notes whose own funding depended on the cash test, so a nominal commitment could not fill the condition on which it depended (FirstMark Horizon Acquisition Corp., 2021, 2022b). Public shareholders held the right to redeem at trust value until the vote, an outside option that kept their participation revocable.

The response used four of the five instruments in Table 4. Non-redemption agreements converted a revocable right into a binding ticket for specified shares and paid for it with additional shares delivered only on closing, a success-contingent transfer addressed to participation (Starry Group Holdings, 2022a). Amended subscriptions required investors to deliver funds into escrow before closing, a cash-funding arrangement, and the credit amendment was a permission: the

lenders agreed not to treat the merger as a change of control, provided defined closing cash reached a separate minimum of $160 million (Starry Group Holdings, 2022b,d). Two further actions changed the game itself. The repricing of the private placement and the Series Z subscription from $10.00 to $7.50 per share changed the terms on which new money participated (Starry Group Holdings, 2022b,c), and the waiver of the $300 million condition changed the threshold, so the closing that followed was formation under amended terms (Starry Group Holdings, 2022e). The transaction closed on March 29 with $176.3 million of gross proceeds (Starry Group Holdings, 2022g).

The record contains no escrowed payment for strategic failure, and no party anchored the combination. The escrowed subscriptions of the new investors were refunds that made their conditional commitments credible without paying anyone for the failure of others, and the $160 million minimum in the credit amendment survived the waiver, so that closing still required a threshold of defined closing cash. The waiver changed the formation structure after the commitments on which it depended had been made. It was executed on March 28, twelve days after the special meeting at which stockholders approved the combination and before which public holders had to elect redemption (FirstMark Horizon Acquisition Corp., 2022c), and with $140 million of new equity committed, so that, in the terms of Section 3, the merger parties acted much as the last mover in an order does, deciding whether the combination formed while knowing approximately what it would hold, and a threshold fixed before redemptions became a choice made after them.

Before that move, public holders were numerous and individually small, so for them the relevant margin was participation, and closing depended on enough of them choosing to remain at the amended terms. A selection problem of the kind Theorem 36 treats could arise only among investors whose subscriptions were conditional on one another, the PIPE and Series Z investors, and the record shows repriced terms and escrow for them but no payment to any of them conditional on others' withdrawal; the lenders' consent was a permission, which the model does not formalize. Public shares form a single class with identical redemption terms, and retention ran through agreements with specified holders rather than through any change in the terms offered to the class as a whole. Although the record establishes selective retention payments, it identifies no provision that would have entitled other holders to the same consideration, so the case shows the first route of Appendix B.9.3, and selective retention outside any clause, at work without establishing that an equal-treatment constraint bound; selective payments funded from outside the class are the cheapest form of retention whether or not such a constraint binds, so their form alone is not diagnostic. The case documents the instruments without measuring their magnitudes, since redemptions do not establish an exogenous improvement in outside options and closing does not establish positive surplus for every participant.

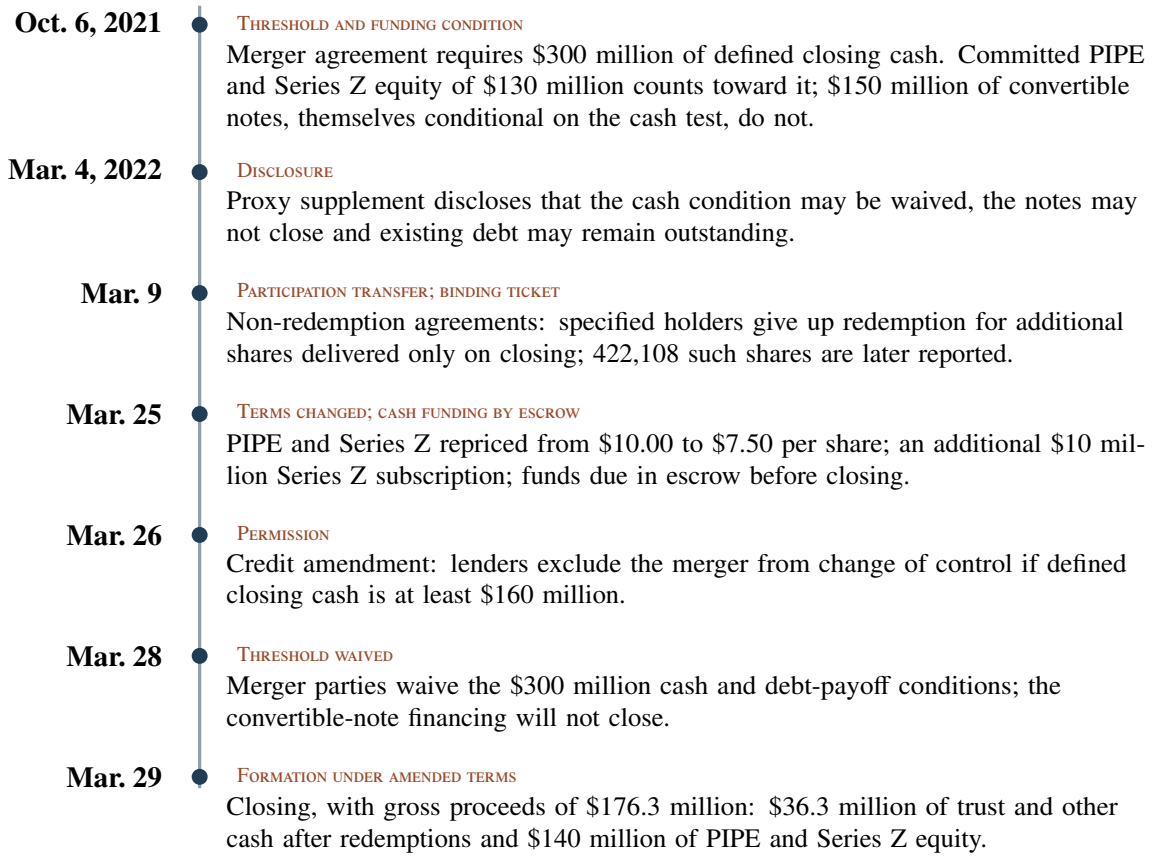


Figure 7: Formation of the FirstMark–Starry combination. Each dated action is labeled by the failure in Table 4 that it addresses, by the change it makes to the game's terms or threshold, or as a disclosure. No action provides an escrowed payment for strategic failure. Sources: merger agreement, proxy supplement, non-redemption form, subscription amendments, credit amendment, merger waiver and first-quarter 2022 report; Internet Appendix V gives the chronology and the cash reconciliation.

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Internet Appendix

I. Transfer feasibility

For a fixed proposed allocation with incremental values g_i and transfer bounds $\underline{T}_i \le T_i \le \overline{T}_i$, a balanced transfer that makes every investor willing exists exactly when

$$\hat{T}_i = \max\{-g_i, \underline{T}_i\} \le \overline{T}_i \quad \text{for every } i, \qquad \sum_i \hat{T}_i \le 0 \le \sum_i \overline{T}_i. \tag{IA.1}$$

This is feasibility of intervals with a zero sum. Real administrative expenses are subtracted from the allocation first, and a dated cash payment requires liquidity on its payment date, not only terminal surplus.

In the Bayesian model, with success transfers satisfying $\underline{T}_i \le T_i \le \overline{T}_i$ and $\sum_i T_i = 0$, strict participation at every type is feasible exactly when the linear program

$$\max_{T,\eta} \ \eta \quad \text{subject to} \quad T_i \ge r_i + \eta, \quad \underline{T}_i \le T_i \le \overline{T}_i, \quad \sum_i T_i = 0 \tag{IA.2}$$

has a strictly positive value, where r_i is the floor (18). The feasible set is compact in T, and a vector strictly above every floor has a positive minimum slack. Here η is slack in success-state dollars, not a common certainty-equivalent gain. Collection of negative transfers, settlement timing and contractual eligibility are further requirements that sure-money translation under CARA does not remove.

II. A finite private-information economy

Tickets are \$80,000, \$100,000, \$120,000 and \$100,000, and the minimum is \$350,000, so the proposed \$400,000 purchase requires all four investors. In thousands of dollars, gross success values are $(14, 13, 16, 12)$, and each investor has two equally likely outside values: $(6, 10)$, $(6, 10)$, $(7, 11)$ and $(4, 12.2)$. Expenses of standing ready are $(0.5, 0.8, 0.6, 1)$, and the common funding condition succeeds with probability 0.9. The high type of investor 4 dislikes success at the initial terms.

The economy of Appendix B.3.3 uses these parameters, and the success transfers, failure payments, reserve, carry, backstop and caps reported there are computed from them. The draw of \$40,000 on a backstop for the full purchase is an expected gross draw in this two-state specification, neither a net credit loss nor a quoted guarantee price, and ordinary subscriptions, binding subscriptions, transfers alone and assurance alone each leave a refusal equilibrium or an unwilling type, so that only the combined contract meets both conditions of Theorem 36.

III. Combining private values and continuous allocation

At $\theta = 1.6$ in the four-investor economy of Appendix B.13.3, freeze the coordinated quantities as a contractual offer, $q = (0.7775, 0.9275, 0.7875, 1.0075)$. Their success increments before transfers are approximately $y = (-0.0716, 6.4634, 4.0834, 7.9994)$, treated here as fixed success-state payments independent of residual wealth and of the funding event, which is what permits the CARA lottery calculation. Add two private outside-value increments per investor, $(0, 0.2)$ for the first and $(0, 0.4)$ for each of the others. Success transfers of 0.4 to the first investor, financed equally by the others, restore strict full-subscription willingness with funding probability 0.9, signing costs of 0.01 and CARA coefficient 0.02 per thousand dollars. Strategic-failure payments of 0.011 to the first three investors and nothing to the fourth secure every signature by Theorem 36. The worst-type certainty-equivalent gains, net of signing costs, are approximately $(0.1056, 5.2944, 3.1735, 6.6573)$, and the reserve for strategic payments is \$33, while the investment requires \$350,000. After acceptance, quantities are contractual tickets rather than revisable choices. The construction shows that a continuous formation loss can be repaired by a contract whose information and funding conditions are checked separately; it does not characterize a simultaneous game in which investors choose quantities and hold private values.

IV. Competing programs

Four investors each choose to stay out, join program X or join program Y. Program X forms with at least three members and pays each member $10 - 3/n_X$, where n_X is its membership. Program Y forms with any membership and pays member i the amount $(14, 14, 1, 1)_i - c_Y/n_Y$. Outside values are zero. With $c_Y = 12$, the equilibria are universal membership of program X, with welfare 37, and profiles in which the first two investors join program Y while the others stay out or one of them joins program X without forming it, with welfare 16; the coordinated optimum is 37. With $c_Y = 2$, the unique equilibrium places every investor in program Y, with payoffs $(13.5, 13.5, 0.5, 0.5)$ and welfare 28, while the coordinated optimum places one of the first two investors alone in program Y and the other three in program X, with welfare 39. A cheaper shared alternative lowers best-equilibrium welfare by 9 and raises coordinated welfare by 2. Allowing each investor to divide his participation between the two programs in halves or in quarters reproduces these extrema.

V. Transaction chronology and cash

The dated record separates the date an instrument was agreed, its public filing date and the period a financial statement measures. The March 4 supplement already described a possible waiver of the cash condition, failure of the convertible financing, waivers by PIPE investors and retention of existing debt, so those possibilities were observable before the March 28 action. The first-quarter accounting was reported after closing and is not assigned to any earlier information set (FirstMark Horizon Acquisition Corp., 2022a,b).

Date	Contract or disclosed event
Oct. 6, 2021	Merger agreement: $300m defined closing-cash condition; $109m PIPE, $21m Series Z and $150m conditional notes. Note proceeds excluded from the definition (sections 1.01 and 9.03(e)).
Feb. 11, 2022	Proxy statement dated; redemption rights and conditional sources-and-uses scenarios disclosed.
Mar. 4	Supplement describes the waiver, no-notes and debt-retention alternative and the interdependence of financing conditions.
Mar. 9	Non-redemption agreements exchange specified redemption rights for additional shares on completion; other shares retain redemption rights.
Mar. 16	Special meeting approves the business combination; redemption elections are due before the vote (FirstMark Horizon Acquisition Corp., 2022c).
Mar. 25	PIPE and original Series Z repriced to $7.50, delivering more shares for the same dollars; additional $10m Series Z subscription; amended escrow deadline.
Mar. 26	Credit amendment excludes the transaction from change of control, conditional on at least $160m of defined closing cash.
Mar. 28	Merger waiver removes the $300m cash and debt-payoff conditions, not the separate credit condition; executed Series Z amendment; first merger.
Mar. 29	Acquisition merger closes; PIPE and Series Z purchases consummated.
After closing	Annual and quarterly reports disclose redemptions, shares issued and the cash and accounting reconciliation.

The first-quarter report records gross proceeds of $176.282 million: $36.282 million of FirstMark trust and other cash and $140 million of PIPE and Series Z financing. Subtracting $12.507 million of transaction costs paid during the period gives $163.775 million of cash-flow net proceeds; subtracting the $18.499 million combined transaction-cost measure gives $157.783 million, the basis of the later registration statement's rounded $157.8 million. Neither equals the $110.930 million equity-accounting entry, which also reflects the separately presented Series Z conversion, warrant liabilities and other adjustments (Starry Group Holdings, 2022g,h). The repricing increased PIPE shares from 10.9 million to 14,533,334 for approximately the same $109 million, and original Series Z shares from 2.1 million to 2.8 million for $21 million; the 422,108 non-redemption shares were reported as issued after closing (Starry Group Holdings, 2022a,b,c,f). No accounting amount measures investor welfare.

VI. Commitment support: computations

Theorem 3 and Propositions 49 and 9. In 200 random formation structures on two to four investors with integer expenses and gains, every schedule of own payments on the grid $\{0,\dots,5\}^n$ that

implements formation in every pure equilibrium pays more than R^* to the investors it assures, and both constructions at $R^* + \varepsilon$ implement formation. In 300 random structures a common escrowed payment implements formation exactly when it exceeds h°, and $\bar{m}\, h^\circ \geq R^*$. In 300 random structures and orders, backward induction forms the investment in every case. In 200 random threshold games, failure payments leave the set of formed pure equilibria unchanged. For schedules that combine own and escrowed payments, an exhaustive search over the grid $\{0, \ldots, 3\}^{2n}$ in random structures on two or three investors, 288,000 schedules in all, confirms that formation is implemented exactly when the investors with $t_i + \beta_i > h_i$, together with at most one other, form the investment, and that an implementing schedule then pays or promises more than R^* to the assured members of an inclusion-minimal forming coalition; the schedule with $t_1 = \beta_2 = 1.1$ under unanimity among three investors with unit expenses is one of them.

Proposition 10, Theorem 13(iii) and Proposition 34. The set of outcomes of pure subgame-perfect equilibria was computed recursively, round by round, for 400 random structures on two to four investors with random assignments to one to three rounds and random signs of $t_i + \beta_i - h_i$; a pure equilibrium exists at every history, and every one forms the investment exactly when the assured investors, together with at most one other investor from each round, form it. In 250 random structures with integer expenses and one to three rounds, the least liability over all assignments and assured sets equals R^*_ℓ, it is zero exactly when some forming coalition has at most ℓ members, and $R^*_1 = R^*$; with four indispensable investors and expenses of \$500, \$800, \$600 and \$1,000 it is \$1,900, \$1,100, \$500 and zero for one to four rounds. In 1,712 random economies with two to four investors in a public order, escrowed payments at most equal to the expenses, and exact rational arithmetic, backward induction at open histories yields willing formation if and only if the condition of Theorem 13(iii) holds, and the expected payment, computed by enumerating willingness profiles, equals $\sum_k p_{\leq k}(1 - p_{>k})\beta_k$; payments just above the lower bound attain the stated infimum. Under constant absolute risk aversion, in 300 random parameterizations, the escrowed payment that makes an early commitment worthwhile costs less in expectation than the own payment that does. In 400 random economies with two to five investors, enumeration of every order confirms that sorting by increasing $p_i h_i/(1 - p_i)$ minimizes the expected expense wasted on failed attempts and that every order wastes strictly less than simultaneous commitments with willing formation.

Section 3: contingent capacity. The script `verify_core.py` checks the results on contingent capacity. In 254 random formation structures on two to four investors with expenses between zero and four, 6,566 randomly drawn payment schemes, including payments to investors who decline, implement formation by iterated strict dominance, and every one has liability above R^*; the escrowed schedule at $R^* + \varepsilon$ implements formation in every structure. In the example of Proposition 7, no action is dominated, the fully mixed case has no solution, and the linear program over correlated equilibria gives a largest probability of failure of $813/815 \approx 0.9975$. For Theorem 5, minimizing liability over dual certificates of unique correlated equilibrium, with weights drawn at random, never yields a value below R^* in 40 random closed cohorts, coming within 0.15 of it in 34, or in 30 random counts with outsiders, and never a value below R_c in 30

random structures; a separate search over 92 structures with locally optimized weights found no value below either bound. These searches illustrate the theorem and do not replace its proof. In the example that follows the theorem, the scheme of liability 6.5 makes formation the unique correlated equilibrium when gains are 20, by an exact dual certificate in rational arithmetic and by the primal linear program, admits a correlated equilibrium that fails for certain when gains are 14, and leaves every action undominated; with gains of 200, a scheme of liability 6.041 makes formation the unique correlated equilibrium; and the count of three seats among four with a payment to an investor who declines has liability 1.1 and a unique correlated equilibrium. For portfolios of two or three investments with two or three investors each, a single capacity slightly above $\max_k R_k^*$, honored by seniority, makes formation of every investment the unique outcome of iterated strict dominance in all 120 cases. Backward iterated dominance was computed history by history for 300 random structures with random assignments to rounds: the staged escrow implements formation in every case with liability within $n\varepsilon$ of $\hat{R}_\ell$, and among 16,952 random schemes contingent on the profile of commitments that implement formation by backward iterated dominance none has liability below $\hat{R}_\ell$. For Proposition 12, the staged escrow's capacity summed over the rounds equals $(n - \ell)h$ for 200 random partitions of indispensable investors. Interim correlated rationalizability was computed for 200 random cohorts of two or three investors with an unwilling type whose loss is 10^{-6}: an escrow paired with a premium or with a fee below the least willing gain yields willing formation as the unique outcome in every case, and the same escrow without a charge in none, while a fee that lies between $(1 - p_{-i})(\beta_i - h_i)$ and $\beta_i - h_i$ but exceeds the willing gain does not; an order with escrowed payments equal to the expenses yields willing formation at every history in 300 random economies whose gains and losses both come within 10^{-9} of zero.

Correlated robustness under risk aversion. The script `verify_book.py` also checks the extension of Theorem 5: the scheme of liability 6.5 that makes formation the unique correlated equilibrium for risk-neutral investors admits a correlated equilibrium that fails for certain under constant absolute risk aversion of 0.05, and in 36 random structures, counts and others, minimizing liability over certificates written in utilities for investors with constant absolute risk aversion never yields a value below R^* for counts or R_c otherwise.

VII. The book: computations

The script `verify_discretion.py` checks the results of Section 4 other than those of Section 4.4, together with Proposition 46, which Internet Appendix IX reports. For Proposition 16, in 400 random books the largest flow computed by linear programming falls short of total requirements by exactly B^*, and widening a permission never raises the shortfall. For Proposition 17, in 458 random books widening permissions enlarges the formation structure and weakly lowers R^*, and in 230 books whose committed capital equals requirements $R^* = \sum_i h_i - \max_i h_i$. For Theorem 18, with two equal projects and reliability of 99 percent the least flexible capital, computed from the exact binomial distribution, is 22, 78, 254 and 811 units for 100, 1,000, 10,000 and 100,000

investors, so that its ratio to $z_{0.995}\sqrt{n}$ rises to 0.996; with three to five projects the bound of part (i) keeps simulated failure below five percent; with shares drawn from a Beta(20,20) distribution and reliability of 95 percent the least flexible share is 0.2380, 0.2338 and 0.2334 for 1,000, 10,000 and 100,000 investors, against the limit of 0.2334; with one mandate in five following a common recommendation it is 0.1708 for 200,000 investors, against the limit of $1/6$; and blocks of 16 investors sharing a mandate raise the requirement by a factor of 3.97. For Theorem 19, with costs uniform on $[0,1]$ the least total cost averages 0.248, 0.253 and 0.248 for two projects and 1,000, 10,000 and 100,000 investors, against the limit of $\tfrac{1}{4}$, and 0.518, 0.501 and 0.493 for three projects, against $\tfrac{1}{2}$; at 10,000 investors pro rata exposure costs 2,500 on average; $\sqrt{n}$ times the expected premium on the crowded project is 0.75, 0.81 and 0.79 for 1,000, 10,000 and 100,000 investors; the cheapest assignment equals the value of the fractional linear program and never exceeds pro rata exposure in 60 random books; with shares drawn from a Beta(20,20) distribution the cost per investor is 0.0053 and 0.0045 at 10,000 and 100,000 investors, against the limit of 0.0049; in 13,200 tests of the uniform-price assignment no misreport of a favored project or a cost gains; a brute-force computation of the Vickrey–Clarke–Groves payments with one place left vacant confirms, for 17,944 payments in random two-project books, that only the over-subscribed project is priced; with a share φ of flexible investors among 20,000 and costs uniform on $[0,1]$, the least cost averages 2.28, 1.01 and 0.51 for φ equal to 0.1, 0.25 and 0.5, against limits of 2.5, 1 and 0.5; with costs drawn from a Beta(2,1) distribution, whose density vanishes at zero, the least cost stays below the bound of part (iii) and grows like $n^{1/4}$; with a cost of at least 0.05 for every move it grows like $\sqrt{n}$; and with $3\sqrt{n}$ flexible investors it rises from 7.7 to 24.4 as the population grows from 10,000 to 100,000. Returning the premium to the investors moved gives an investor who favors the uncrowded project a mean gain of 0.92 from declaring the crowded one, and under the uniform price a truthful finite cost is never worse than a narrow mandate in 2,000 tests. Enlarging the set of willing investors weakly lowers $R^{*}(\Gamma, \mathcal{W})$ in all 182 random books in which the smaller set can finance the program. For Proposition 20, paying investors who are moved gives an indifferent investor a mean gain of 0.170 from declaring the crowded project, and the uniform-price assignment gives none. For Proposition 28, exact posteriors at every imbalance up to 3,000 confirm the band, whose width falls from 127.7 to 1.3 as ε rises from 0.001 to 0.1 when $v^{+}=1$ and $v^{-}=-0.6$.

VIII. Forming the book and the distribution of the cost of discretion: computations

The script `verify_book.py` checks Theorem 22, Corollaries 23, 41, 27 and 42, Propositions 24, 25 and 26 and part (v) of Theorem 19, together with the extension of Theorem 5 to risk-averse investors, which Internet Appendix VI reports. Exact rational arithmetic confirms $\mathbb{E}[C_n] = n/(4(n+1))$ for every even n up to 80 with uniform costs and two equal projects. With 200,000 investors and uniform costs, the realized cost has mean 0.250, median 0.109 and 99th percentile 1.651, against the limits 0.25, 0.114 and 1.659 of $Z^{2}/4$, and $\mathbb{E}[D_n]/\sqrt{n}$ and $\sqrt{n}\,\mathbb{E}[\varpi_n]$ are 0.394

and 0.781, against $\mathbb{E}|Z|/2 = 0.399$ and $\mathbb{E}|Z| = 0.798$. Under a retained veto with uniform costs and margins, the shortfall divided by $\sqrt{n}$ has mean 0.379 and 95th percentile 0.976 at 40,000 investors with the premium retained, against the limits 0.399 and 0.980 of $|Z|/2$, and mean 0.196 and 95th percentile 0.510 with it returned, against 0.199 and 0.490 of $|Z|/4$; with the premium retained, the probability of no shortfall falls from 0.039 to 0.014 as the population grows from 4,000 to 40,000. With costs whose density at zero is 2 and margins whose density at zero is 3, the shortfall scales with $\gamma/2 = 0.75$. With replacement capital $a\sqrt{n}$ and a equal to 0.5, 1 and 1.5, the simulated probabilities of closing at 40,000 investors are 0.693, 0.957 and 1.000, against the limits 0.683, 0.954 and 0.997 of $\mathbb{P}(|Z| \le 2a)$. Under binding enrollment with the premium retained, the expected burden, computed exactly from the distribution of the over-subscription, gives $\sqrt{n}\, v_n^*$ equal to 0.3950, 0.3977, 0.3985 and 0.3988 for 4,000, 40,000, 400,000 and four million investors, against the limit $1/\sqrt{2\pi} = 0.3989$, and the number declining concentrates. With the premium returned, the net burdens sum to the realized cost of moves in every draw, so that each investor's expected net burden is $\mathbb{E}[C_n]/n$, and the probability of closing at critical load, $(1 - 1/(4(n+1)))^n$, is 0.7788 at 100,000 investors, against $e^{-\mu_v} = e^{-1/4}$. At 99 percent reliability, with the premium retained and costs and margins uniform, the exact replacement capital under binding enrollment is 55, 153 and 446 units for ten thousand, a hundred thousand and a million investors, against limiting requirements of 129, 407 and 1,288 under a retained veto, ratios of 0.427, 0.376 and 0.346 that fall toward the limit of 0.310. In 61,500 misreports with the premium returned, no gain exceeds half the larger of the two gaps between adjacent reports of the others at the margin, and under a retained veto the investor whose report sets the price gains in expectation by shading it. For part (v), exact rational arithmetic confirms the step of the proof that rewrites the bound on an investor's expected net burden by the order-statistic recurrence, for K up to 29. Computing each type's expected net burden exactly by numerical integration over order-statistic distributions, with every other investor enrolled and uniform costs, n times the bound, which equals n times the largest expected net burden and is attained by the investors with the highest costs, is 0.480, 0.490 and 0.495 for 400, 1,600 and 6,400 investors; when the others' enrollment rate is one for costs below one half and one half above, so that the enrolled costs have densities $4/3$ and $2/3$, it is 0.360, 0.368 and 0.371. No type's expected net burden exceeds the bound, and at 100 investors the exact computation agrees with simulation of the assignment, -0.0123 for a cost of 0.05 and 0.0046 for a cost of one.

For Proposition 24, in the four-investor example with reports $(0.9, 0.8, 0.1, -0.5)$ the third investor gains 0.30 by reporting 0.7 when the premium is returned in equal shares and nothing under the leave-one-out rebate, and in 73,200 misreports of favored project and cost across random books of four to eight investors no misreport gains under the leave-one-out rebate, while equal shares allow gains up to 0.450. In random books of 10 to 200 investors the rebates never exceed the premium, the net burdens sum to the realized cost of moves plus the premium not returned, and that premium equals $K(K+1)/n$ times the gap between the price and the next lower cost on the crowded side, as a direct computation of every investor's leave-one-out price confirms. With uniform costs its exact expectation is 0.4461, 0.4824, 0.4944 and 0.4982 for 200, 2,000, 20,000

and 200,000 investors, against the limit $1/(2g_c(0)) = 0.5$; at 20,000 investors its median is 0.331 and the probability that it exceeds 0.5 is 0.359, against 0.347 and $e^{-1} = 0.368$ for the exponential limit, and with costs whose density at zero is 2 its mean is 0.249, against 0.25. The probability of closing at critical load is 0.4736 at 100,000 investors, against $e^{-3/4} = 0.4724$, and two or three additional invitations raise the limit to 0.9595 and 0.9927. With investors who know their own types, n times the largest expected net burden under the leave-one-out rebate is 0.960, 0.980, 0.990 and 0.995 for 400, 1,600, 6,400 and 25,600 investors, and at 100 investors simulation gives 0.926 against an exact 0.920.

For Corollary 41, with uniform costs and 20,000 investors the total net burden under the leave-one-out rebate has mean 0.760, median 0.605 and 99th percentile 2.835, against 0.75, 0.591 and 2.836 for $\chi^2_3/4$, and a Kolmogorov–Smirnov test does not reject that limit (p-value 0.66). For Proposition 25, in 500 cases drawn from random books of 6 to 20 investors, under the uniform price and under the leave-one-out rebate, an investor is placed in A exactly when his report exceeds the price he faces, his transfer is constant on each side of that price, and the two transfers differ by it. With costs and margins uniform, the inequality of the proof holds investor by investor: over 40,100 pairs of the price faced, ϖ°, and the transfer on the uncrowded side, the expected withdrawal less half the expected transfer collected never falls below $(1 - \varpi^\circ)\varpi^\circ/4$, so that, by the argument of the proof, every mechanism unsubsidized in expectation that makes the efficient assignment with dominant-strategy incentives has $\mathbb{E}[Q_n] \geq \frac{n}{4}\,\mathbb{E}[|\varpi^\circ_1|(1 - |\varpi^\circ_1|)]$, which is 6.06 at 1,000 investors. In a family of such mechanisms that pay every investor θ times the price he faces, less a fee τ_0/n, so that an investor on the side crowded without him bears $(1 - \theta)|\varpi^\circ_i|$, no member with nonnegative expected revenue loses fewer than the 6.06 investors that the bound allows; the uniform price with the premium retained loses 12.77, and the member with $\theta = \frac{1}{2}$ and no fee loses 6.19 but runs an expected deficit of 0.63. Under the leave-one-out rebate, $\mathbb{E}[Q_n]/\sqrt{n}$ is 0.205 at 4,000 investors and 0.196 at 40,000, against the limit $\gamma\,\mathbb{E}|Z|/4 = 0.1995$, and the transfers collected from investors who accept average 0.361 at both sizes, so that no fee is needed; with margins whose density at zero is six times that of costs they average -0.25 without a fee and 0.74 with $\tau_0 = 1$. For the expected-externality mechanism, at 40,000 investors $\sqrt{n}\,\bar{\varpi}^\circ_n(0)$ is 0.3989 and $n\,\mathbb{E}[\bar{\varpi}^\circ_n(c_1)]$ is 0.502, with c_1 drawn from G_c, against $1/\sqrt{2\pi}$ and $\frac{1}{2}$. With truthful reports under a retained veto, the conditional mean of the shortfall given costs and favored projects, $\sum_i G_v(b_i^+)$, where b_i is investor i's net burden, averages 0.636, 0.649 and 0.656 at 4,000, 40,000 and 400,000 investors, with standard errors 0.006, 0.008 and 0.017, against the limit 0.659; the shortfall itself averages 0.635 and 0.701 at 4,000 and 40,000 investors, with standard errors 0.016 and 0.023, and is zero with probability 0.547 and 0.511, against 0.535. An investor whose cost is one half gains by reporting a cost of zero when his margin is half of $\underline{v}_n$, by 1.8×10^{-3} and 5.4×10^{-4} at 4,000 and 40,000 investors, and loses when it is one and a half times $\underline{v}_n$, while $\sqrt{n}\,\underline{v}_n$ is 0.383 and 0.394, against $1/\sqrt{2\pi} = 0.399$. If every investor whose cost is at least one half and whose margin is below $\underline{v}_n$ reports a cost of zero, the shortfall at 40,000 investors averages 19.0, against 0.70 with truthful reports.

For Proposition 26, with costs and margins uniform, so that $\lambda_v = \frac{1}{2}$ and the bound of Proposition 25(i) is 0.1995 $\sqrt{n}$, the parity device loses no investor in any state in which the number favoring the crowded project is odd, and its cash deficit in those states, with mean 0.511 and 0.470 at 4,000 and 40,000 investors, is not distinguishable from $Z^2/2$ by a Kolmogorov–Smirnov test (p-values 0.39 and 0.42); the probability of no withdrawal is 0.532 and 0.486, the 99th percentile of its deficit is 4.30 and 3.80, its expected shortfall divided by $\sqrt{n}$ is 0.201 and 0.211, and the capacity $Q_n + \Delta_n/2$ that suffices with probability 0.4 is 0.4 units at both sizes, while that sufficing with probability 0.6 is 0.118 $\sqrt{n}$ and 0.165 $\sqrt{n}$, against a limit of 0.127 $\sqrt{n}$. The share of states in which a mechanism is right about all but five percent of its investors is 0.496 for the parity device at 4,000 investors and zero for the leave-one-out rebate, for the mechanisms that pay every investor θ times the price he faces with θ equal to 0, $\frac{1}{2}$ and 1, and for the rare-state mechanism described in the text, which, with a charge of one half on every other investor when the first investor reports a cost below 0.0319, closes without replacement capital with probability 0.976 at 40,000 investors and collects 157 in expectation, while its expected shortfall is 2.36 $\sqrt{n}$ and the median of its cash deficit 0.321 $\sqrt{n}$. In the family that pays every investor θ times the price he faces less a fee τ_0/n, the expected places plus half the expected cash, divided by $\sqrt{n}$, ranges from 0.194 to 0.419 at 4,000 investors and never falls more than a standard error below the finite-sample bound 0.1955 $\sqrt{n}$, the bound of the proof before the limit is taken; with $\theta = 1$ and no fee the whole of it is cash. For Corollary 42, in 300 books of 2,000 investors the expected-externality transfers sum to zero and the net burdens sum to the realized cost of moves in every draw. For Corollary 27, exact rational arithmetic confirms that with uniform costs the expected cash paid to the investors moved, $\mathbb{E}[D_n X_{(D_n+1)}]$, equals $n/(2(n+1))$ for every even n up to 80, twice the expected cost of the moves; in 17,343 misreports of cost across books of four to ten investors no misreport gains; and at 20,000 investors no investor withdraws in any of 1,500 books, the cash paid averages 0.476 against the limit 0.5, and a Kolmogorov–Smirnov test does not distinguish its distribution from $Z^2/2$ (p-value 0.69).

IX. Delegation, records and exposure profiles: computations

Theorem 29. In 250 randomly generated environments with two accounts, two positions per account, three or four states, a trivial, two binary and a full experiment with random costs, and a random target that uses the full experiment, the linear program for own-account compensation was solved with and without nonnegativity, and the dual program searching for a cheaper replicating plan was solved separately; the two agree in all 500 cases, 196 of them infeasible. In a further 250 environments the program for strict implementation agrees with the condition of part (ii) in every case, 102 of them strictly implementable. In a one-account example with four equally likely states, an action right in the first two states and another right in the last two, and experiments costing 0, 1 and 2, the cheapest plan replicating a target that uses the middle experiment costs exactly as much as the target, so the target can be made optimal but not the manager’s unique choice.

The four-investor economy. Enumeration gives 120 feasible allocations and 29 in the menu of separate permissions. Under own-account payments of $5/4$ for each account that holds a program and whose increment equals 20, the manager's values are 3 for investigating both bits, $5/2$ without investigation and $13/10$ for one bit, on the menu of separate permissions and on all 120 allocations; at a payment of 1 the first two coincide at 2. Under the schedule tied to allocations, the uninformed manager obtains 7 from D against 6 from investigating, and the schedule violates (21) in every state. Joint compensation paid only when all four accounts are right deters every deviation on $\{0, P, Q\}$ at an expected cost of 4.

Joint terms as economies. In 253 random two-account environments in which the target is implementable both on the full menu and on the menu of its own allocations, the least nonnegative own-account compensation is never higher on the smaller menu and is strictly lower in 37. For 300 random pairs of joint record laws on two accounts with two or three records each, the linear programs for the least joint and the least own-account compensation equal $c/(1 - \varrho_{\text{joint}})$ and $c/(1 - \varrho_{\text{own}})$, and $\varrho_{\text{joint}} \le \varrho_{\text{own}}$ in every case. With $q_0 = 1/2$ and $q_1 = 3/5$ over four concentrated accounts, the two costs are 6 and $1296/671$; with $q_1 = 11/20$ over 20 accounts they are 11 and 1.17.

Proposition 45. In the four-account example, each investor values investigation at 2; a lone concentrated account must pay 6 with nonnegative payments and 1 without that restriction, and an account holding a quarter of each bet must pay $1296/671$.

For Proposition 46, in 30 random economies profiles spanned by the leading singular vectors attain a loss of half the sum of the squared singular values beyond the k-th, and none of 50 randomly drawn subspaces of the same dimension does better; the optimal number of profiles falls from 4 to 3 to 1 as the cost of a record rises, and to 2 with four classes; and background covariances proportional to one vector give optimal holdings of rank two.

X. Numerical methods

Every numerical example and figure is reproduced by exact rational enumeration of pure equilibria, entry deviations and ties in the threshold games, by analytic optimization with direct checks of unilateral deviations in the continuous games, and by enumeration of finite Bayesian games, including mixed equilibria and correlated types. The same checks are applied to randomly generated binary, continuous and Bayesian economies; they test the theorems at their boundaries and do not substitute for the proofs. The replication archive, containing code, inputs and results for every number and figure, accompanies the paper.